

LENGTH-TWO REPRESENTATIONS OF QUANTUM AFFINE SUPERALGEBRAS AND BAXTER OPERATORS

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ABSTRACT. Associated to quantum affine general linear Lie superalgebras are two families of short exact sequences of representations whose first and third terms are irreducible: the Baxter TQ relations involving infinite-dimensional representations; the extended T-systems of Kirillov–Reshetikhin modules. We make use of these representations over the *full* quantum affine superalgebra to define Baxter operators as transfer matrices for the quantum integrable model and to deduce Bethe Ansatz Equations, under genericity conditions.

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INTRODUCTION

Fix $\mathfrak{g} := \mathfrak{gl}(M|N)$ a general linear Lie superalgebra and q a non-zero complex number which is not a root of unity. Let $U_q(\widehat{\mathfrak{g}})$ be the associated quantum affine superalgebra [46]. This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a q -deformation of the universal enveloping algebra of the affine Lie superalgebra of central charge zero $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

$U_q(\widehat{\mathfrak{g}})$ contains a commutative Cartan subalgebra. It defines a monoidal category $\mathcal{O}_{\mathfrak{g}}$ of representations of $U_q(\widehat{\mathfrak{g}})$, by imposing the finite-dimensionality of weight spaces and the cone restriction of weights, as for Kac–Moody algebras [35] but without integrability condition [30, 39]. In [51] we proposed the generic asymptotic representations in $\mathcal{O}_{\mathfrak{g}}$, denoted by $\mathcal{W}_{c,a}^{(i)}$ with i being a Dynkin node of \mathfrak{g} and $c, a \in \mathbb{C}^{\times}$ (referred to as spin and spectral parameters). They were obtained as limits of certain irreducible finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules, the Kirillov–Reshetikhin modules, by modifying a construction of Hernandez–Jimbo [32] in categories $\mathcal{O}_{\mathfrak{b}}$ of Borel subalgebras of quantum affine algebras.

The purpose of this paper is two-fold: to produce in the Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{g}})$ of the category $\mathcal{O}_{\mathfrak{g}}$ the mutation-like identities of the form

$$(1) \quad [\mathcal{T}][\mathcal{W}_{c,a}^{(i)}] = [\mathcal{W}'] + [\mathcal{W}'']$$

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where \mathcal{T} is a $U_q(\widehat{\mathfrak{g}})$ -module in category $\mathcal{O}_{\mathfrak{g}}$ and $\mathcal{W}', \mathcal{W}''$ are suitable tensor products of the $\mathcal{W}_{d,b}^{(j)}$; to define for the quantum integrable system attached to $U_q(\widehat{\mathfrak{g}})$, XXZ spin chain, the Baxter operators as transfer matrices of the $\mathcal{W}_{c,a}^{(i)}$ and interpret (1) as *Baxter's TQ relation*, a functional relation of transfer matrices.

Large part of this paper is devoted to the proof of (1). It concerns the representation theory of $U_q(\widehat{\mathfrak{g}})$, which was less studied compared to quantum affine algebras (e.g. [12, 38]) mainly due to the smallness of Weyl group symmetry. Quantum affine superalgebras and Yangians are related to the integrability of important supersymmetric models like deformed Hubbard model and anti de Sitter/conformal field theory correspondences (see [8, 7] and references therein).

1. *Baxter operators.* In an exactly solvable model a common problem is to find the spectrum of a family $T(z)$ of commuting endomorphisms of a vector space V depending on a complex spectral parameter z , called transfer matrices. The Bethe Ansatz method, initiated by H. Bethe, gives explicit eigenvectors and eigenfunctions of $T(z)$ in terms of solutions to a system of algebraic equations, the Bethe Ansatz equations (BAE). Typical examples are the Heisenberg spin chain and the ice model. In [2], for the 6-vertex model R. Baxter related $T(z)$ to another family of commuting endomorphisms $Q(z)$ on V by the relation:

$$(2) \quad T(z) = a(z) \frac{Q(zq^2)}{Q(z)} + d(z) \frac{Q(zq^{-2})}{Q(z)}.$$

Here $a(z), d(z)$ are scalar functions and q is the parameter of the model. $Q(z)$ is a polynomial in z , called Baxter operator. The cancellation of poles at the right-hand side leads to BAE for the roots of $Q(z)$. Similar operatorial equation also holds for the 8-vertex model [2], where the Bethe Ansatz method fails.

The transfer matrix $T(z)$ has an interpretation in terms of representations of a quantum group \mathbf{U} . Let $\mathcal{R}(z) \in \mathbf{U}^{\otimes 2}$ be the universal R-matrix with spectral parameter z and let V, W be two representations of \mathbf{U} . Then $t_W(z) := \text{tr}_W(\mathcal{R}(z)_{W \otimes V})$ defines a commuting family of endomorphisms on V , thanks to the quasi-triangularity of $(\mathbf{U}, \mathcal{R}(z))$. As a first example, for the 6-vertex model, $\mathbf{U} = U_q(\mathfrak{sl}_2)$ is a quantum affine algebra, $W = \mathbb{C}^2$ is a vector representation and $V = (\mathbb{C}^2)^{\otimes \ell}$. Similarly, for the face-type Andrews–Baxter–Forrester model, $\mathbf{U} = E_{\tau, \eta}(\mathfrak{sl}_2)$ is an elliptic quantum group [19, 20], and $W = \mathbb{C}^2, V = (\mathbb{C}^2)^{\otimes \ell}$.

In the affine case, $Q(z)$ can be written as a transfer matrix. This was first observed for $U_q(\widehat{\mathfrak{sl}}_2)$ by Bazhanov–Lukyanov–Zamolodchikov [3] and recently extended by Frenkel–Hernandez [23] to arbitrary non-twisted quantum affine algebras. The main feature is that the first tensor factor of $\mathcal{R}(z)$ lies in a Borel subalgebra \mathfrak{b} of \mathbf{U} , so that $t_W(z)$ makes sense for W a \mathfrak{b} -module. In $Q(z) = t_{\mathcal{L}}(z)$, $\mathcal{L} = L_{i,a}^+$ is a positive *prefundamental module* over \mathfrak{b} , for i a Dynkin node of the underlying finite-dimensional simple Lie algebra and $a \in \mathbb{C}^\times$. The $L_{i,a}^+$ are irreducible objects of a category $\mathcal{O}_{\mathfrak{b}}$ of \mathfrak{b} -modules introduced by Hernandez–Jimbo [32]. Equation (2) comes from an identity involving $[L_{i,a}^+]$ in the Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{b}})$ [23].

In the elliptic case, the triangular structure of $\mathcal{R}(z)$ is less clear as there is not yet a formulation of a Borel subalgebra. Still the eigenvalues of $T(z)$ are of the form (2) by a Bethe Ansatz in [20]. In a joint work with G. Felder [21], we were able to construct elliptic Baxter operator $Q(z)$ for $E_{\tau, \eta}(\mathfrak{sl}_2)$ as a transfer matrix of *asymptotic representations* over the full elliptic quantum group.

Then a natural question is whether the Baxter operators can always be realized from representations of the full quantum group (of type Yangian, affine, or elliptic). Inspired by [21], in the present paper we provide a partial answer for the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, based on the asymptotic representations $\mathcal{W}_{c,a}^{(i)}$.

The key property of the $\mathcal{W}_{c,a}^{(i)}$ is the separation of variable identity (Lemma 9.5):

$$(3) \quad [\mathcal{W}_{c,1}^{(i)}][\mathcal{W}_{1,a^2}^{(i)}] = [\mathcal{W}_{ca,a^2}^{(i)}][\mathcal{W}_{a^{-1},1}^{(i)}] \in K_0(\mathcal{O}_{\mathfrak{g}}).$$

This categorifies $\frac{c-zc^{-1}}{1-z} \frac{1-za^2}{1-za^2} = \frac{ca-zc^{-1}a}{1-za^2} \frac{a^{-1}-za}{1-z}$ upon identification of $\mathcal{W}_{c,a}^{(i)}$ with the rational function $\frac{c-zac^{-1}}{1-za}$ in z at the i -th position. Such rational functions appear naturally as the highest (loop) weights of irreducible objects in category $\mathcal{O}_{\mathfrak{g}}$ [30, 39]. If $m \in \mathbb{Z}_{>0}$, then $\mathcal{W}_{q_i^m, aq_i}^{(i)}$ has a finite-dimensional irreducible socle $W_{m,a}^{(i)}$, the Kirillov–Reshetikhin module; here $q_i \in \{q^{\pm 1}\}$ depends on the Dynkin node i . Reciprocally $\mathcal{W}_{c,aq_i}^{(i)}$ is an asymptotic limit “ $q_i^\infty = c$ ” of the $W_{m,a}^{(i)}$ with $m \rightarrow \infty$.

Set $Q_i(u)$ to be the transfer matrix of $\mathcal{W}_{u,1}^{(i)}$ evaluated at 1 (Definition 9.6). Strictly speaking $Q_i(z)$ is not a transfer matrix, yet the ratio $\frac{Q_i(zc)}{Q_i(z)}$ for fixed $c \in \mathbb{C}^\times$ is indeed a ratio of transfer matrices, thanks to Equation (3). If V is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module, then $t_V(z^{-2})$ is a sum of monomials of the $d(z) \frac{Q_i(zac)}{Q_i(za)}$ where the $d(z)$ are scalar functions, the number of terms being $\dim V$. This forms a generalized Baxter TQ relation in the spirit of [23].

2. Baxter TQ relations. For higher rank quantum groups, the generalized TQ relations are different from Equation (2) due to the absence of two-dimensional irreducible representations, and they are not enough to derive BAE. The recent works [17] on toroidal \mathfrak{gl}_1 and [33, 24, 18] on a non-twisted quantum affine algebra \mathbf{U} show that Equation (2) still holds, by taking $T(z)$ as the transfer matrix of certain irreducible \mathfrak{b} -module $N_{i,a}^+$ in category $\mathcal{O}_{\mathfrak{b}}$, for i a Dynkin node i and $a \in \mathbb{C}^\times$. Indeed $[N_{i,a}^+][L_{i,a}^+] = [\mathcal{L}'] + [\mathcal{L}'']$ in the Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{b}})$ where $\mathcal{L}', \mathcal{L}''$ are tensor products of positive prefundamental modules. These identities are interpreted in [33] as Fomin–Zelevinsky cluster mutations. In general, the \mathfrak{b} -module structures on $N_{i,a}^+, L_{i,a}^+$ can not be extended to \mathbf{U} , and they are infinite-dimensional.

The main result of the paper (Theorem 5.5) is that for the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, there are similar mutation-like relations in $K_0(\mathcal{O}_{\mathfrak{g}})$: the $L_{i,a}^+$ are replaced by asymptotic representations $\mathcal{W}_{c,a}^{(i)}$ and the $N_{i,a}^+$ by $U_q(\widehat{\mathfrak{g}})$ -modules $\mathcal{N}_{c,a}^{(i)}, M_{c,a}^{(i)}$. The pair $(\mathcal{W}_{c,a}^{(i)}, M_{c,a}^{(i)})$ enables us to write down BAE for the Baxter operators $Q_i(z)$ under polynomiality assumptions; see Equation (9.39).

The proof of Theorem 5.5 requires an extension of $\mathcal{O}_{\mathfrak{g}}$ to a category \mathcal{O} of representations of a Borel subalgebra $Y_q(\mathfrak{g})$ of $U_q(\widehat{\mathfrak{g}})$. Category \mathcal{O} contains positive/negative prefundamental modules $L_{i,a}^\pm$. The asymptotic limits [32, 51] connect the $U_q(\widehat{\mathfrak{g}})$ -modules $\mathcal{W}_{c,a}^{(i)}$ with the $Y_q(\mathfrak{g})$ -modules $L_{i,a}^-$. TQ relations are derived for the $L_{i,a}^\pm$ (Theorem 3.3 and Corollary 5.1) similarly as [33, 18]. These relations are translated into the $\mathcal{W}_{c,a}^{(i)}$ by a duality functor \mathbb{D} of categories \mathcal{O} (Lemma 1.8). A crucial step is the asymptotic construction of $\mathcal{N}_{c,a}^{(i)}$ in Section 7. The logic goes as follows (for simplicity we drop the Dynkin node i and spin/spectral parameters a, c):

$$N^+ L^+ (3.27) \xrightarrow[\text{duality}]{\mathbb{D}} N^- L^- (5.28) \xrightarrow[\text{replacement}]{\text{asymptotic}} \mathcal{NW} (5.29) \xrightarrow[\text{duality}]{\mathbb{D}} M\mathcal{W} (5.30).$$

Along the way, we improve the representation theory of $U_q(\widehat{\mathfrak{g}})$ in [47, 49, 50, 51]:

- (i) the q -character theory of Frenkel–Reshetikhin [26], which describes the Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{g}})$ in terms of rational functions (Theorem 2.4);
- (ii) a criteria (Theorem 6.1) for a tensor product of Kirillov–Reshetikhin modules $W_{m,a}^{(i)}$ to admit an irreducible head, and Theorem 3.4 on the difference of $W_{m,aq_i^2}^{(i)} \otimes W_{m+s,a}^{(i)}$ and $W_{m+s+1,aq_i^2}^{(i)} \otimes W_{m-1,a}^{(i)}$ for m, s positive integers.

When $s = 0$, Theorem 3.4 reduces to the T-system [40, 29, 42].

3. We expect that the $\mathcal{W}_{c,a}^{(i)}, \mathcal{N}_{c,a}^{(i)}, M_{c,a}^{(i)}$ have analogy in elliptic quantum groups, based on twistor theory relating quantum affine algebras to elliptic quantum groups [34, 28, 37]. This would result in elliptic Baxter operators and BAE. The rank one case has been studied in [21].

It is possible to adapt the arguments to the case of Yangians (not necessarily of type A) in view of [28]. One could avoid *degenerate Yangian* [4, 5, 27], whose prefundamental representations lead to Baxter operators but do not carry natural action of the ordinary Yangian. [21, Appendix] discussed the \mathfrak{gl}_2 case. The Yangian of centrally extended $\mathfrak{psl}(2|2)$ [7] is of special interest in AdS/CFT. We do not know of any representation category \mathcal{O} with well-behaved highest weight theory, yet there are limit constructions of infinite-dimensional representations [1].

For twisted quantum affine algebras \mathbf{U} , there are conjectural TQ relations in category $\mathcal{O}_{\mathbf{b}}$ of \mathbf{b} -modules [24]. One may ask for such relations in terms of \mathbf{U} -modules. This is interesting from another point of view: the correspondence between twisted quantum affine algebras and non-twisted quantum affine superalgebras [52, 16]. A typical example is the equivalence [16] of categories \mathcal{O}_{int} of integrable representations over $U_q(A_{2n}^{(2)})$ and $U_q(\mathfrak{osp}(1|2n))$. Let us mention an earlier work of Z. Tsuboi [43] on BAE for Lie superalgebras of type BD, the representation theory meaning of which is still to be understood. For this purpose, the work [45] on Drinfeld second realization of quantum affine superalgebras should be helpful.

The paper is structured as follows. In Section 1 we review the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ and its Borel subalgebra $Y_q(\widehat{\mathfrak{g}})$, discuss the categories $\mathcal{O}, \mathcal{O}'$, and construct a duality functor $\mathbb{D} : \mathcal{O} \rightarrow \mathcal{O}'$. Section 5 presents the main result, \mathcal{NW} and MW identities in the Grothendieck ring $K_0(\mathcal{O})$ for the $U_q(\widehat{\mathfrak{g}})$ -modules $X_{c,a}^{(i)}$ with $X \in \{\mathcal{W}, \mathcal{N}, M\}$, and reduces MW to $\mathbb{D}(\mathcal{NW})$. In Section 9, for the $U_q(\widehat{\mathfrak{g}})$ XXZ spin chain, we construct Baxter operators from the $\mathcal{W}_{c,a}^{(i)}$ and derive Bethe Ansatz Equations from the MW identity.

The two basics ingredients are: the q -character formulas in terms of Young tableaux, proved in Section 2; cyclicity of tensor products of Kirillov–Reshetikhin modules studied in Section 6. The q -characters already lead to TQ relations of positive prefundamental modules in Sections 3–4. The proof of the \mathcal{NW} identity is completed in Section 7 upon realizing $\mathcal{N}_{c,a}^{(i)}$ as a proper asymptotic limit.

The extended T-systems of Kirillov–Reshetikhin modules are proved in Section 8. Although they are not needed in the proof of the main theorem, we include them here as applications of q -characters and cyclicity.

1. BASICS ON QUANTUM AFFINE SUPERALGEBRAS

Fix $M, N \in \mathbb{Z}_{>0}$. In this section we collect basic facts on the quantum affine superalgebra associated with the general linear Lie superalgebra $\mathfrak{g} := \mathfrak{gl}(M|N)$ and its representations. The main references are [49, 50, 51], some of whose results are modified to be coherent with the non-graded quantum affine algebras.

Set $\kappa := M + N$, $I := \{1, 2, \dots, \kappa\}$ and $I_0 := I \setminus \{\kappa\}$. Let \mathbb{Z}_2 denote the ring $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. The *weight lattice* \mathbf{P} is the abelian group freely generated by the ϵ_i for $i \in I$. Let $[\cdot]$ be the morphism of additive groups $\mathbf{P} \rightarrow \mathbb{Z}_2$ such that

$$|\epsilon_1| = |\epsilon_2| = \dots = |\epsilon_M| = \bar{0}, \quad |\epsilon_{M+1}| = |\epsilon_{M+2}| = \dots = |\epsilon_{M+N}| = \bar{1}.$$

\mathbf{P} is equipped with a symmetric bilinear form $(\cdot, \cdot) : \mathbf{P} \times \mathbf{P} \rightarrow \mathbb{Z}$,

$$(\epsilon_i, \epsilon_j) = \delta_{ij}(-1)^{|\epsilon_i|} \quad \text{where } (-1)^{\bar{0}} := 1, (-1)^{\bar{1}} := -1.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $i \in I_0$, and the *root lattice* \mathbf{Q} to be the subgroup of \mathbf{P} generated by the α_i . Set $q_l := q^{(\epsilon_l, \epsilon_l)}$ and $q_{ij} := q^{(\alpha_i, \alpha_j)}$ for $i, j \in I_0$ and $l \in I$.

If W is a vector superspace and $w \in W$ is a \mathbb{Z}_2 -homogeneous vector, then by abuse of language let $|w| \in \mathbb{Z}_2$ denote the parity of w . (It is not to be confused with the absolute value $|n|$ of an integer n .)

Let \mathbf{V} be the vector superspace with basis $(v_i)_{i \in I}$ and parity $|v_i| := |\epsilon_i|$. Define the elementary matrices $E_{ij} \in \text{End}(\mathbf{V})$ by $E_{ij}v_k = \delta_{jk}v_i$ for $i, j, k \in I$. They form a basis of the vector superspace $\text{End}(\mathbf{V})$ and $|E_{ij}| = |\epsilon_i| + |\epsilon_j|$.

1.1. Quantum superalgebras. Recall the Perk–Schultz matrix [41]

$$\begin{aligned} R(z, w) = & \sum_{i \in I} (zq_i - wq_i^{-1}) E_{ii} \otimes E_{ii} + (z - w) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\ & + z \sum_{i < j} (q_i - q_i^{-1}) E_{ji} \otimes E_{ij} + w \sum_{i < j} (q_j - q_j^{-1}) E_{ij} \otimes E_{ji}. \end{aligned}$$

It is well-known that $R(z, w)$ satisfies the quantum Yang–Baxter equation:

$$R_{12}(z_1, z_2) R_{13}(z_1, z_3) R_{23}(z_2, z_3) = R_{23}(z_2, z_3) R_{13}(z_1, z_3) R_{12}(z_1, z_2) \in \text{End}(\mathbf{V})^{\otimes 3}.$$

The convention for the tensor subscripts is as usual. Let $n \geq 2$ and A_1, A_2, \dots, A_n be unital superalgebras. Let $1 \leq i < j \leq n$. If $x \in A_i$ and $y \in A_j$, then

$$(x \otimes y)_{ij} := (\otimes_{k=1}^{i-1} 1_{A_k}) \otimes x \otimes (\otimes_{k=i+1}^{j-1} 1_{A_k}) \otimes y \otimes (\otimes_{k=j+1}^n 1_{A_k}) \in \otimes_{k=1}^n A_k.$$

Now we can define the quantum affine superalgebra associated to \mathfrak{g} .

Definition 1.1. [49, §3.1] $U_q(\widehat{\mathfrak{g}})$ is the superalgebra with presentation:

- (R1) RTT-generators $s_{ij}^{(n)}, t_{ij}^{(n)}$ of parity $|\epsilon_i| + |\epsilon_j|$ for $i, j \in I$ and $n \in \mathbb{Z}_{\geq 0}$;
- (R2) RTT-relations in $U_q(\widehat{\mathfrak{g}}) \otimes (\text{End}(\mathbf{V})^{\otimes 2})[[z, z^{-1}, w, w^{-1}]]$

$$\begin{aligned} R_{23}(z, w) T_{12}(z) T_{13}(w) &= T_{13}(w) T_{12}(z) R_{23}(z, w), \\ R_{23}(z, w) S_{12}(z) S_{13}(w) &= S_{13}(w) S_{12}(z) R_{23}(z, w), \\ R_{23}(z, w) T_{12}(z) S_{13}(w) &= S_{13}(w) T_{12}(z) R_{23}(z, w); \end{aligned}$$

- (R3) $t_{ij}^{(0)} = s_{ji}^{(0)} = 0$ and $s_{kk}^{(0)} t_{kk}^{(0)} = 1$ for $i, j, k \in I$ and $i < j$.

$T(z) \in U_q(\widehat{\mathfrak{g}}) \otimes \text{End}(\mathbf{V})[[z^{-1}]]$ and $S(z) \in U_q(\widehat{\mathfrak{g}}) \otimes \text{End}(\mathbf{V})[[z]]$ are power series

$$\begin{aligned} T(z) &= \sum_{ij} t_{ij}(z) \otimes E_{ij}, \quad t_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} t_{ij}^{(n)} z^{-n}, \\ S(z) &= \sum_{ij} s_{ij}(z) \otimes E_{ij}, \quad s_{ij}(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} s_{ij}^{(n)} z^n. \end{aligned}$$

The *Borel subalgebra* $Y_q(\mathfrak{g})$, also called q -Yangian, is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s_{ij}^{(n)}$ and $(s_{ii}^{(0)})^{-1}$. The finite-type quantum supergroup $U_q(\mathfrak{g})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s_{ij}^{(0)}$ and $t_{ij}^{(0)}$.

$U_q(\widehat{\mathfrak{g}})$ has a Hopf superalgebra structure with counit $\varepsilon : U_q(\widehat{\mathfrak{g}}) \rightarrow \mathbb{C}$ defined by $\varepsilon(s_{ij}^{(n)}) = \varepsilon(t_{ij}^{(n)}) = \delta_{ij} \delta_{n0}$, and coproduct $\Delta : U_q(\widehat{\mathfrak{g}}) \rightarrow U_q(\widehat{\mathfrak{g}})^{\otimes 2}$:

$$\Delta(s_{ij}^{(n)}) = \sum_{m=0}^n \sum_{k \in I} \epsilon_{ijk} s_{ik}^{(m)} \otimes s_{kj}^{(n-m)}, \quad \Delta(t_{ij}^{(n)}) = \sum_{m=0}^n \sum_{k \in I} \epsilon_{ijk} t_{ik}^{(m)} \otimes t_{kj}^{(n-m)}.$$

Here $\epsilon_{ijk} := (-1)^{|E_{ik}||E_{kj}|}$. The antipode $\mathbb{S} : U_q(\widehat{\mathfrak{g}}) \rightarrow U_q(\widehat{\mathfrak{g}})$ is determined by

$$(\mathbb{S} \otimes \text{Id})(S(z)) = S(z)^{-1}, \quad (\mathbb{S} \otimes \text{Id})(T(z)) = T(z)^{-1}.$$

$S(z)^{-1}$ and $T(z)^{-1}$ are well-defined owing to Definition 1.1 (R3). Notice that $Y_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are sub-Hopf-superalgebras of $U_q(\widehat{\mathfrak{g}})$.

We shall need $U_{q^{-1}}(\widehat{\mathfrak{g}})$, whose RTT generators are denoted by $\overline{s}_{ij}^{(n)}, \overline{t}_{ij}^{(n)}$.

Recall the following are isomorphisms of Hopf superalgebras ($a \in \mathbb{C}^\times$):

$$(1.1) \quad \Phi_a : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)}, \quad t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)},$$

$$(1.2) \quad \Psi : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\text{cop}}, \quad s_{ij}^{(n)} \mapsto \varepsilon_{ji} t_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji} s_{ji}^{(n)},$$

$$(1.3) \quad h : U_{q^{-1}}(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\text{cop}}, \quad \overline{S}(z) \mapsto S(z)^{-1}, \quad \overline{T}(z) \mapsto T(z)^{-1}.$$

Here $\varepsilon_{ij} := (-1)^{|\epsilon_i|+|\epsilon_i||\epsilon_j|}$ and A^{cop} of a Hopf superalgebra A takes the same underlying superalgebra but the twisted coproduct $\Delta^{\text{cop}} := c_{A,A}\Delta$, with $c_{A,A} : x \otimes y \mapsto (-1)^{|x||y|}y \otimes x$ the graded permutation, and antipode \mathbb{S}^{-1} . There are superalgebra morphisms for $p(z) \in \mathbb{C}[[z]]^\times$, $p_1(z) \in \mathbb{C}[[z^{-1}]]^\times$ with $p(0)p_1(\infty) = 1$:

$$(1.4) \quad \text{ev}_a^+ : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\mathfrak{g}), \quad s_{ij}(z) \mapsto \frac{s_{ij}^{(0)} - zat_{ij}^{(0)}}{1 - za}, \quad t_{ij}(z) \mapsto \frac{t_{ij}^{(0)} - z^{-1}a^{-1}s_{ij}^{(0)}}{1 - z^{-1}a^{-1}},$$

$$(1.5) \quad \phi_{[p,p_1]} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s_{ij}(z) \mapsto p(z)s_{ij}(z), \quad t_{ij}(z) \mapsto p_1(z)t_{ij}(z).$$

$\Phi_a, h, \text{ev}_a^+, \phi_{[p,p_1]}$ restrict to $Y_q(\mathfrak{g})$ or $Y_q(\mathfrak{g}')$, denoted by $\Phi_a, h, \text{ev}_a^+, \phi_p$. Let $\overline{\text{ev}}_a^+ : U_{q^{-1}}(\widehat{\mathfrak{g}}) \longrightarrow U_{q^{-1}}(\mathfrak{g})$ be the corresponding morphisms when replacing q by q^{-1} . This gives rise to (notice that $h(U_{q^{-1}}(\mathfrak{g})) = U_q(\mathfrak{g})$):

$$(1.6) \quad \text{ev}_a^- : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\mathfrak{g}), \quad \text{ev}_a^- = h \circ \overline{\text{ev}}_a^+ \circ h^{-1}.$$

$U_q(\widehat{\mathfrak{g}})$ is \mathbf{Q} -graded: $x \in U_q(\widehat{\mathfrak{g}})$ is of weight $\lambda \in \mathbf{Q}$ if $s_{ii}^{(0)}x = q^{(\lambda, \epsilon_i)}xs_{ii}^{(0)}$ for all $i \in I$. For example $s_{ij}^{(n)}$ and $t_{ij}^{(n)}$ are of weight $\epsilon_i - \epsilon_j$ [49, (3.14)]. Let $U_q(\widehat{\mathfrak{g}})_\lambda$ be the weight space of weight λ . The \mathbf{Q} -grading restricts to $Y_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$.

We recall the *Drinfeld second realization* of $U_q(\widehat{\mathfrak{g}})$ from [49, §3.1.4]. Write

$$\begin{cases} S(z) = (\sum_{i < j} e_{ij}^+(z) \otimes E_{ij} + 1)(\sum_l K_l^+(z) \otimes E_{ll})(\sum_{i < j} f_{ji}^+(z) \otimes E_{ji} + 1), \\ T(z) = (\sum_{i < j} e_{ij}^-(z) \otimes E_{ij} + 1)(\sum_l K_l^-(z) \otimes E_{ll})(\sum_{i < j} f_{ji}^-(z) \otimes E_{ji} + 1), \end{cases}$$

as invertible power series in $z^{\pm 1}$ over $U_q(\widehat{\mathfrak{g}}) \otimes \text{End}(\mathbf{V})$; the subscripts $i, j, l \in I$. Notice that $K_\kappa^+(z) = s_{\kappa\kappa}(z)$. For $i \in I_0$, $j \in I$ let us define τ_i, θ_j :

$$(1.7) \quad \tau_i := q^{M-N+1-i} \quad \text{for } 1 \leq i \leq M, \quad \tau_{M+l} := q^{l+1-N} \quad \text{for } 1 \leq l < N,$$

$$(1.8) \quad \theta_j := q^{2(M-N+1-j)} \quad \text{for } 1 \leq j \leq M, \quad \theta_{M+l} := q^{2(l-N)} \quad \text{for } 1 \leq l \leq N.$$

The Drinfeld loop generators are defined by generating series: let $i \in I_0$,

$$\begin{aligned} x_i^+(z) &= \sum_{n \in \mathbb{Z}} x_{i,n}^+ z^n := \frac{e_{i,i+1}^+(z\tau_i) - e_{i,i+1}^-(z\tau_i)}{q_i - q_i^{-1}} \in U_q(\widehat{\mathfrak{g}})[[z, z^{-1}]], \\ x_i^-(z) &= \sum_{n \in \mathbb{Z}} x_{i,n}^- z^n := \frac{f_{i+1,i}^-(z\tau_i) - f_{i+1,i}^+(z\tau_i)}{q_i^{-1} - q_i} \in U_q(\widehat{\mathfrak{g}})[[z, z^{-1}]], \\ \phi_i^\pm(z) &= \sum_{n \geq 0} \phi_{i,\pm n}^\pm z^{\pm n} := K_i^\pm(z\tau_i)K_{i+1}^\pm(z\tau_i)^{-1} \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]]. \end{aligned}$$

Remark 1.2. In [49, §3.1.4] a different Gauss decomposition of $S(z), T(z)$ was considered (f always ahead of e). If $\overline{X}_i^\pm(z), \overline{K}_l^\pm(z)$ with $i \in I_0$, $l \in I$ denote the Drinfeld generating series of $U_{q^{-1}}(\widehat{\mathfrak{g}})$ in *loc. cit.*, then

$$h(\overline{K}_l^\pm(z)) = K_l^\pm(z)^{-1}, \quad h(\overline{X}_i^\pm(z)) = \pm(q_i^{-1} - q_i)x_i^\pm(z).$$

One can rewrite [49, Theorem 3.5] in terms of the $x_i^\pm(z), \phi_i^\pm(z), K_l^\pm(z)$. For $i, j \in I_0$, $l, l' \in I$ and $\eta, \eta' \in \{\pm\}$ we have: (recall $q_{ij} = q^{(\alpha_i, \alpha_j)}$)

$$K_l^\eta(z)K_{l'}^{\eta'}(w) = K_{l'}^{\eta'}(w)K_l^\eta(z), \quad K_l^+(0)K_l^-(\infty) = 1,$$

$$\begin{aligned}
K_{M+N}^\eta(z)x_i^\pm(w) &= \left(\frac{zq^{-1} - wq}{z - w} \right)^{\pm\delta_{i+1, M+N}} x_i^\pm(w) K_{M+N}^\eta(z), \\
\phi_i^\eta(z)x_j^\pm(w) &= \frac{z - wq_{ij}^{\pm 1}}{zq_{ij}^{\pm 1} - w} x_j^\pm(w) \phi_i^\eta(z), \\
[x_i^+(z), x_j^-(w)] &= \delta_{ij} \frac{\phi_i^+(z) - \phi_i^-(w)}{q_i - q_i^{-1}} \delta\left(\frac{z}{w}\right), \\
(zq_{ij}^{\pm 1} - w)x_i^\pm(z)x_j^\pm(w) &= (z - wq_{ij}^{\pm 1})x_j^\pm(w)x_i^\pm(z) \quad \text{if } (i, j) \neq (M, M), \\
[x_i^\pm(z_1), [x_i^\pm(z_2), x_j^\pm(w)]_q]_{q^{-1}} + \{z_1 \leftrightarrow z_2\} &= 0 \quad \text{if } (i \neq M, |j - i| = 1), \\
x_M^\pm(z)x_M^\pm(w) &= -x_M^\pm(w)x_M^\pm(z), \quad x_i^\pm(z)x_j^\pm(w) = x_j^\pm(w)x_i^\pm(z) \quad \text{if } |i - j| > 1,
\end{aligned}$$

together with the degree 4 oscillator relation when $M, N > 1$:

$$[[[x_{M-1}^\pm(u), x_M^\pm(z_1)]_q, x_{M+1}^\pm(v)]_{q^{-1}}, x_M^\pm(z_2)] + \{z_1 \leftrightarrow z_2\} = 0.$$

Here $[x, y]_a := xy - a(-1)^{|x||y|}yx$ for $x, y \in U_q(\widehat{\mathfrak{g}})$ and $a \in \mathbb{C}$. These relations are coherent with the Drinfeld second realization of quantum affine algebras (e.g. [30, §3.2]) and superalgebras [46, Theorem 8.5.1]. For $i \in I_0 \setminus \{M\}$, the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $(x_{i,n}^\pm, \phi_{i,n}^\pm)_{n \in \mathbb{Z}}$ is a quotient algebra of $U_{q_i}(\widehat{\mathfrak{sl}}_2)$.

Let $\mathbf{Q}^+ := \oplus_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathbf{P}$ and $\mathbf{Q}^- := -\mathbf{Q}^+$. By [49, Proposition 3.6]:

$$(1.9) \quad \Delta(K_i^\pm(z)) \in K_i^\pm(z) \otimes K_i^\pm(z) + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{-\alpha} \otimes U_q(\widehat{\mathfrak{g}})_\alpha [[z^{\pm 1}]],$$

$$(1.10) \quad \Delta(x_i^+(z)) \in x_i^+(z) \otimes 1 + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{\alpha_i - \alpha} \otimes U_q(\widehat{\mathfrak{g}})_\alpha [[z, z^{-1}]],$$

$$(1.11) \quad \Delta(x_i^-(z)) \in 1 \otimes x_i^-(z) + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{-\alpha} \otimes U_q(\widehat{\mathfrak{g}})_{\alpha - \alpha_i} [[z, z^{-1}]].$$

The coproduct shares the same triangular property with [26, Lemma 1].

1.2. Category \mathcal{O} . We first recall the notion of weights from [51, §6]. Define

$$\mathfrak{P} := (\mathbb{C}^\times)^I \times \mathbb{Z}_2, \quad \widehat{\mathfrak{P}} := (\mathbb{C}[[z]]^\times)^I \times \mathbb{Z}_2.$$

The multiplicative group structure on $\mathbb{C}^\times, \mathbb{C}[[z]]^\times$ and the *additive* group structure on the ring \mathbb{Z}_2 make $\mathfrak{P}, \widehat{\mathfrak{P}}$ into multiplicative abelian groups. \mathfrak{P} is naturally a subgroup of $\widehat{\mathfrak{P}}$, and $\mathbb{C}[[z]]^\times \longrightarrow \mathbb{C}^\times, f(z) \mapsto f(0)$ induces a projection $\varpi : \widehat{\mathfrak{P}} \longrightarrow \mathfrak{P}$. There is an injective homomorphism of abelian groups (see also [18, §3.1])

$$(1.12) \quad q^\lambda : \mathbf{P} \longrightarrow \mathfrak{P}, \quad \lambda \mapsto q^\lambda := ((q^{(\epsilon_i, \lambda)})_{i \in I}; |\lambda|).$$

Let V be a $Y_q(\mathfrak{g})$ -module. For $p = ((p_i)_{i \in I}; s) \in \mathfrak{P}$, define

$$V_p := \{v \in V_s \mid s_{ii}^{(0)} v = p_i v \text{ for } i \in I\}.$$

If $V_p \neq 0$, then p is called a *weight* of V , and V_p the weight space of weight p . Let $\text{wt}(V)$ denote the set of weights of V . Notice that $s_{ij}^{(n)} V_p \subseteq V_{q^{\epsilon_i - \epsilon_j} p}$ for $p \in \text{wt}(V)$. Similarly, for $\widehat{p} = ((p_i(z))_{i \in I}; s) \in \widehat{\mathfrak{P}}$ define

$$V_{\widehat{p}} := \{v \in V_s \mid \exists d \in \mathbb{Z}_{>0} \text{ such that } (K_i^+(z) - p_i(z))^d v = 0 \text{ for } i \in I\}.$$

If $V_{\widehat{p}} \neq 0$, then \widehat{p} is an ℓ -weight of V , and $V_{\widehat{p}}$ the ℓ -weight space of ℓ -weight \widehat{p} . Let $\text{wt}_\ell(V)$ be the set of ℓ -weights of V .

Example 1.3. Let $\widehat{p} = ((p_i(z))_{i \in I}; s) \in \widehat{\mathfrak{P}}$ be such that $\frac{p_i(z)}{p_j(z)} \in \mathbb{C}^\times$ for $i \neq j$. There is a representation of $Y_q(\mathfrak{g})$ on the one-dimensional vector superspace $\mathbb{C} = \mathbb{C}1_s$ of parity $s = |1_s|$, defined by $s_{ij}(z)1_s = \delta_{ij}p_i(z)1_s$. Let $\mathbb{C}^{\widehat{p}}$ denote this $Y_q(\mathfrak{g})$ -module. We have $\{\widehat{p}\} = \text{wt}_\ell(\mathbb{C}^{\widehat{p}})$ and $\{\varpi(\widehat{p})\} = \text{wt}(\mathbb{C}^{\widehat{p}})$.

Definition 1.4. [51, §6.2] A $Y_q(\mathfrak{g})$ -module V is in category \mathcal{O} if:

- (i) V has a weight space decomposition $V = \bigoplus_{p \in \mathfrak{P}} V_p$;
- (ii) $\dim V_p < \infty$ for all $p \in \mathfrak{P}$;
- (iii) there exist $\mu_1, \mu_2, \dots, \mu_d \in \mathfrak{P}$ such that $\text{wt}(V) \subseteq \bigcup_{j=1}^d (\mu_j q^{\mathbb{Q}^-})$.

Let \mathcal{E}_ℓ be the set of \mathbb{Z} -linear combinations (possibly infinite) $\sum_{\hat{p}} n_{\hat{p}} \hat{p}$ over the $\hat{p} \in \hat{\mathfrak{P}}$ such that $\bigoplus_{\hat{p}} (\mathbb{C}^{\varpi(\hat{p})})^{\oplus |n_{\hat{p}}|} \in \mathcal{O}$. Make \mathcal{E}_ℓ into a ring: addition is the usual one of formal sums; multiplication is induced by that of $\hat{\mathfrak{P}}$. A module V in category \mathcal{O} has an ℓ -weight decomposition, leading to the q -character and the classical character

$$(1.13) \quad \chi_q(V) = \sum_{\hat{p} \in \text{wt}_\ell(V)} \dim(V_{\hat{p}}) \hat{p}, \quad \chi(V) = \sum_{p \in \text{wt}(V)} \dim(V_p) p \in \mathcal{E}_\ell.$$

Following Example 1.3 we have $\chi_q(\mathbb{C}^{\hat{p}}) = \hat{p}$ and $\chi(\mathbb{C}^{\hat{p}}) = \varpi(\hat{p})$.

Let V be a $Y_q(\mathfrak{g})$ -module in category \mathcal{O} . A non-zero \mathbb{Z}_2 -homogeneous vector $v \in V$ is called a *highest ℓ -weight* vector if it is of ℓ -weight $\hat{p} = ((p_i(z))_{i \in I}; s)$ and annihilated by the $s_{ij}^{(n)}$ for $i < j$. Necessarily $K_i^+(z)v = p_i(z)v$. Call V a highest ℓ -weight module if it is generated by a highest ℓ -weight vector v . In this case, v is unique up to scalar product and its ℓ -weight is called the highest ℓ -weight of V . Lowest ℓ -weight vector/module can be defined in a similar way.

Let \mathbf{R} be the subset of $\hat{\mathfrak{P}}$ consisting of the $\hat{p} = ((p_i(z))_{i \in I}; s)$ such that $\frac{p_i(z)}{p_{i+1}(z)}$ is the Taylor expansion at $z = 0$ of a rational function for $i \in I_0$. In addition to \hat{p} , we will also let $\mathbf{n}, \mathbf{n}(z), \mathbf{m}, \mathbf{m}(z), \dots$ denote elements of \mathbf{R} .

Lemma 1.5. [51, Lemma 6.7 & Proposition 6.8] *Let $\hat{p} \in \mathbf{R}$.*

- (1) *If $0 \neq V$ is in category \mathcal{O} , then V contains a highest ℓ -weight vector, and the ℓ -weight of any highest ℓ -weight vector of V belongs to \mathbf{R} .*
- (2) *In category \mathcal{O} there exists an irreducible highest ℓ -weight module $L(\hat{p})$ of highest ℓ -weight \hat{p} . This module is unique up to isomorphism.*
- (3) *$\dim L(\hat{p}) = 1$ if and only if $\frac{p_i(z)}{p_{i+1}(z)} \in \mathbb{C}^\times$ for $i \in I_0$.*
- (4) *$\dim L(\hat{p}) < \infty$ if and only if for $i \in I_0 \setminus \{M\}$ there exist $P_i(z) \in 1 + z\mathbb{C}[z]$ and $c_i \in \mathbb{C}^\times$ such that $\frac{p_i(z)}{p_{i+1}(z)} = c_i \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$.*
- (5) *$L(\hat{p})$ can be extended to a $U_q(\hat{\mathfrak{g}})$ -module if and only if $\frac{p_i(z)}{p_{i+1}(z)}$ is a product of the $c \frac{1-zac^{-2}}{1-za}$ with $a, c \in \mathbb{C}^\times$ for $i \in I_0$.*

Based on (5), let \mathbf{R}_U be the subset of \mathbf{R} consisting of $\hat{p} = ((p_i(z))_{i \in I}; s)$ such that for $i \in I$, the rational function $p_i(z)$ is a product of the $c \frac{1-zac^{-2}}{1-za}$ with $a, c \in \mathbb{C}^\times$. For $\hat{p} \in \mathbf{R}_U$, the $Y_q(\mathfrak{g})$ -module $L(\hat{p})$ is extended *uniquely* to a $U_q(\hat{\mathfrak{g}})$ -module by

$$K_i^+(z)v = p_i(z)v = K_i^-(z)v \quad \text{for } i \in I.$$

Here v is a highest ℓ -weight vector, and in the second identity one views $p_i(z) \in \mathbb{C}[[z^{-1}]]$ by taking the its Taylor expansion of at $z = \infty$. We continue to let $L(\hat{p})$ denote the irreducible $U_q(\hat{\mathfrak{g}})$ -module thus obtained for $\hat{p} \in \mathbf{R}_U$.

Define the completed Grothendieck group $K_0(\mathcal{O})$: elements are \mathbb{Z} -linear combinations (possibly infinite) $\sum_{\hat{p}} n_{\hat{p}} [L(\hat{p})]$ over the $\hat{p} \in \mathbf{R}$ such that $\bigoplus_{\hat{p}} n_{\hat{p}} \hat{p} \in \mathcal{O}$; addition is the usual one of formal sums. For $V \in \mathcal{O}$, the multiplicity $m_{L(\hat{p}), V} \in \mathbb{Z}_{\geq 0}$ of $L(\hat{p})$ in V is well-defined due to Definition 1.4, as in the case of Kac–Moody algebras [35, §9.6]. Set $[V] = \sum_{\hat{p}} m_{L(\hat{p}), V} [L(\hat{p})] \in K_0(\mathcal{O})$. Make $K_0(\mathcal{O})$ into a ring by $[V][W] := [V \otimes W]$. Equation (1.13) extends uniquely to morphisms of additive groups $\chi_q : K_0(\mathcal{O}) \rightarrow \mathcal{E}_\ell$ and $\chi : K_0(\mathcal{O}) \rightarrow \mathcal{E}_\ell$.

Proposition 1.6. χ, χ_q are ring morphisms and χ_q is injective.

Proof. That χ, χ_q are ring morphisms is [51, Lemma 6.3]. The injectivity is proved in the usual way [26, Theorem 3]. \square

For $\mathbf{n} \in \mathbf{R}$ define the *normalized q -character* $\tilde{\chi}_q(L(\mathbf{n})) := \mathbf{n}^{-1} \chi_q(L(\mathbf{n}))$.

Definition 1.7. Let $i \in I_0$, $a \in \mathbb{C}^\times$ and $m \in \mathbb{Z}_{>0}$.

- (i) $\Psi_{i,a} := ((p_j(z))_{j \in I}; \bar{0}) \in \mathbf{R}$ is such that: $\frac{p_j(z)}{p_{j+1}(z)} = (1 - za\tau_j^{-1})^{\delta_{ji}}$ for $j \in I_0$; if $i \leq M$ then $p_\kappa(z) = 1$; if $i > M$ then $p_1(z) = 1$.
- (ii) $Y_{i,a} := q^{\varpi_i} \Psi_{i,aq_i^{-1}} \Psi_{i,aq_i}^{-1} \in \mathbf{R}_U$ where $\varpi_i = \epsilon_1 + \epsilon_2 + \dots + \epsilon_i$ if $i \leq M$ and $\varpi_i = -(\epsilon_{i+1} + \epsilon_{i+2} + \dots + \epsilon_\kappa)$ if $i > M$.
- (iii) $\varpi_{m,a}^{(i)} := Y_{i,a} Y_{i,aq_i^{-2}} \dots Y_{i,aq_i^{2-2m}} \in \mathbf{R}_U$.
- (iv) $L_{i,a}^\pm := L(\Psi_{i,a}^{\pm 1})$ are positive/negative *prefundamental modules* over $Y_q(\mathfrak{g})$, and $W_{m,a}^{(i)} := L(\varpi_{m,a}^{(i)})$ *Kirillov–Reshetikhin module* over $U_q(\widehat{\mathfrak{g}})$.

For $V, W \in \mathcal{O}$, write $V \simeq W$ if there is a one-dimensional module $D \in \mathcal{O}$ such that $V \cong W \otimes D$ as $Y_q(\mathfrak{g})$ -modules. For $\mathbf{m}, \mathbf{n} \in \mathbf{R}$ write $\mathbf{m} \equiv \mathbf{n}$ if $L(\mathbf{m}) \simeq L(\mathbf{n})$; in case $\mathbf{m}\mathbf{n}^{-1} = (h(z)^I; \bar{0})$ for some $h(z) \in \mathbb{C}(z)$, we simply write $\mathbf{m} = h(z)\mathbf{n}$.

1.3. Category \mathcal{O}' . Following [50, §1], let $\mathfrak{gl}(N|M) =: \mathfrak{g}'$ and define the Hopf superalgebras $U_q(\widehat{\mathfrak{g}}'), Y_q(\mathfrak{g}'), U_q(\mathfrak{g}')$ in the same way as for $U_q(\widehat{\mathfrak{g}}), Y_q(\mathfrak{g}), U_q(\mathfrak{g})$ in Section 1.1, except that M, N are interchanged. We start from the same weight/root lattices \mathbf{P}, \mathbf{Q} and $\widehat{\mathfrak{P}}, \widehat{\mathfrak{P}}$ but with different parity map $|\cdot|' : \mathbf{P} \rightarrow \mathbb{Z}_2$:

$$|\epsilon_1|' = |\epsilon_2|' = \dots = |\epsilon_N|' = \bar{0}, \quad |\epsilon_{N+1}|' = |\epsilon_{N+2}|' = \dots = |\epsilon_{N+M}|' = \bar{1},$$

bilinear form $(\epsilon_i, \epsilon_j)' = \delta_{ij}(-1)^{|\epsilon_i|'}$, and embedding $q'^\lambda := ((q^{(\lambda, \epsilon_i)'})_{i \in I}; |\lambda|')$ of \mathbf{P} in $\widehat{\mathfrak{P}}$. One defines category \mathcal{O}' of $Y_q(\mathfrak{g}')$ -modules as in Section 1.2. Let us summarize the modifications of notations related to \mathfrak{g}' :

(1.14)	$\mathfrak{g}, U_q(\mathfrak{g}), Y_q(\mathfrak{g}), U_q(\widehat{\mathfrak{g}})$	$\mathfrak{g}', U_q(\mathfrak{g}'), Y_q(\mathfrak{g}'), U_q(\widehat{\mathfrak{g}}')$	algebras
	$s_{ij}^{(n)}, t_{ij}^{(n)}, q_i, q_{ij}, \tau_i, \theta_j$	$s_{ij}'^{(n)}, t_{ij}'^{(n)}, q_i', q_{ij}', \tau_i', \theta_j'$	RTT
	$x_i^\pm(z), K_i^\pm(z), \phi_i^\pm(z)$	$x_i'^\pm(z), K_i'^\pm(z), \phi_i'^\pm(z)$	currents
	$\mathcal{O}, \mathbf{R}, \mathbf{R}_U, \chi, \chi_q, \simeq, \equiv$	$\mathcal{O}', \mathbf{R}, \mathbf{R}_U, \chi', \chi_q', \simeq, \equiv$	characters
	$\Psi_{i,a}, Y_{i,a}, \varpi_{m,a}^{(i)}, \varpi_i$	$\Psi_{i,a}', Y_{i,a}', \varpi_{m,a}'^{(i)}, \varpi_i'$	ℓ -weights
	$L(\widehat{p}), \mathbb{C}^{\widehat{p}}, L_{i,a}^\pm, W_{m,a}^{(i)}$	$L'(\widehat{p}), \mathbb{C}^{\widehat{p}}, L_{i,a}'^\pm, W_{m,a}'^{(i)}$	modules

In case $M = N$ one can simply remove all the primes in the table.

For $i, j \in I$, set $\widehat{i} := \kappa + 1 - i$ and $\varepsilon_{ij}' := (-1)^{|\epsilon_i|' + |\epsilon_i|'|\epsilon_j|'}$. Then

$$(1.15) \quad f : U_q(\widehat{\mathfrak{g}}') \rightarrow U_q(\widehat{\mathfrak{g}})^{\text{cop}}, \quad s_{ij}'^{(n)} \mapsto \varepsilon_{ji}' s_{\widehat{j}\widehat{i}}^{(n)}, \quad t_{ij}'^{(n)} \mapsto \varepsilon_{ji}' t_{\widehat{j}\widehat{i}}^{(n)}.$$

defines a Hopf superalgebra isomorphism. Let $\bar{f} : U_{q^{-1}}(\widehat{\mathfrak{g}}') \rightarrow U_{q^{-1}}(\widehat{\mathfrak{g}})^{\text{cop}}$ and $h' : U_{q^{-1}}(\widehat{\mathfrak{g}}') \rightarrow U_q(\mathfrak{g}')^{\text{cop}}$ be analogs of Equations (1.15) and (1.3). They induce

$$(1.16) \quad f^- : U_q(\widehat{\mathfrak{g}}') \rightarrow U_q(\widehat{\mathfrak{g}})^{\text{cop}}, \quad f^- := h \circ \bar{f} \circ h'^{-1}$$

a Hopf superalgebra isomorphism which restricts to $f^- : Y_q(\mathfrak{g}') \rightarrow Y_q(\mathfrak{g})$.

Lemma 1.8. *The pullback $(f^-)^*$ induces an anti-equivalence of monoidal categories $\mathbb{D} : \mathcal{O} \rightarrow \mathcal{O}'$. If $\widehat{p} = ((p_i(z))_{i \in I}; s) \in \mathbf{R}$, then as $Y_q(\mathfrak{g}')$ -modules*

$$\mathbb{D}(L(\widehat{p})) \cong L'((p_i(z))_{i \in I}; s).$$

In particular, $\mathbb{D}(L_{i,a}^\pm) \simeq L_{M+N-i, aq^{N-M}}'^\mp$ for $1 \leq i < M + N$.

\mathbb{D} can be viewed as a categorification of the duality function D of Grothendieck rings in [33, Theorem 5.17]. We shall make extensive use of it: to change the signature of the $L_{i,a}^\pm$; to pass from Dynkin nodes $i \leq M$ to $i \geq M$.

Proof. Let V be a $Y_q(\mathfrak{g})$ -module in category \mathcal{O} . If $p \in \mathfrak{P}$, then $V_p = ((f^-)^*V)_{p'}$ where $p' = ((p_i)_{i \in I}; s)$, and so $V_{pq^{n\alpha_i}} = ((f^-)^*V)_{p'q'^{n\alpha_{\kappa-i}}}$ for $i \in I_0$ and $n \in \mathbb{Z}$. This implies that $(f^-)^*V \in \mathcal{O}'$. The first statement is now clear.

Let $V = L(\widehat{p})$ and let $v \in V$ be a highest ℓ -weight vector. In h^*V we have

$$\overline{K}_l^+(z)h^*v = p_l(z)^{-1}h^*v, \quad \overline{s}_{ij}(z)h^*v = 0 \quad \text{for } i, j, l \in I, \ i < j.$$

From the Gauss decomposition of $h^{-1}(S(z))$ it follows that $\overline{s}_{il}(z)h^*v = \overline{K}_l^+(z)h^*v$. Similar identities hold when replacing h^*v by \overline{f}^*h^*v . This implies:

$$\begin{aligned} \overline{K}_i^{'+}(z)\overline{f}^*h^*v &= \overline{s}_{ii}'(z)\overline{f}^*h^*v = \overline{f}^*\left(\overline{s}_{ii}'(z)h^*v\right) = \overline{f}^*\left(\overline{K}_i^+(z)h^*v\right) = p_i(z)^{-1}\overline{f}^*h^*v, \\ K_i^{'+}(z)(f^-)^*v &= K_i^{'+}(z)(h'^{-1})^*\overline{f}^*h^*v = (h'^{-1})^*\left(\overline{K}_i^{'+}(z)^{-1}\overline{f}^*h^*v\right) = p_i(z)(f^-)^*v, \end{aligned}$$

leading to the second statement; here the $\overline{s}_{ii}'(z)$, $\overline{K}_i^{'+}(z)$ denote the RTT generators and Drinfeld generators of $U_{q^{-1}}(\widehat{\mathfrak{g}}')$ arising from [49]; see Remark 1.2. The last statement is a comparison of highest ℓ -weights based on $\tau'_{M+N-i} = \tau_i q^{N-M}$. \square

2. TABLEAU-SUM FORMULAS OF q -CHARACTERS

We compute $\chi_q(L(\mathbf{m}))$ for $\mathbf{m} \in \mathbf{R}_U$ coming from Young diagrams.

To $p = ((p_i)_{i \in I}; s) \in \mathfrak{P}$ is associated a unique irreducible $U_q(\mathfrak{g})$ -module $V_q(p)$, which is generated by a vector v of parity s subject to the following relations:

$$s_{ii}^{(0)}v = p_i v, \quad s_{jk}^{(0)}v = 0 \quad \text{for } i, j, k \in I, \ j < k.$$

For $\lambda \in \mathbf{P}$, set $V_q(\lambda) := V_q(q^\lambda)$. (It was denoted by $V(\lambda)$ in [9, §3.3].)

Definition 2.1. [9, §4.2] \mathcal{P} is the set of $\lambda = \sum_i \lambda_i \epsilon_i \in \mathbf{P}$ such that:

- (S1) $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M \geq 0$ and $\lambda_{M+1} \geq \lambda_{M+2} \geq \dots \geq \lambda_\kappa \geq 0$;
- (S2) if $\lambda_{M+j} > 0$ for some $1 \leq j \leq N$, then $\lambda_M \geq j$.

To $\lambda \in \mathcal{P}$ we attach a subset Y_+^λ of $\mathbb{Z}_{>0}^2$ consisting of (k, l) such that: $l \leq \lambda_k$ for $1 \leq k \leq M$; if $k > M$ then $l \leq N$ and $k \leq M + \lambda_{M+l}$. Let $\mathcal{B}_+(\lambda)$ be the set of functions $T : Y_+^\lambda \rightarrow I$ such that:

- (T1) $T(k, l) \leq T(k', l')$ if $k \leq k'$, $l \leq l'$ and $(k, l), (k', l') \in Y_+^\lambda$;
- (T2) $T(k, l) < T(k+1, l)$ if $(k, l), (k+1, l) \in Y_+^\lambda$ and $T(k, l) \leq M$;
- (T3) $T(k, l) < T(k, l+1)$ if $(k, l), (k, l+1) \in Y_+^\lambda$ and $T(k, l) > M$.

Let $Y_-^\lambda = -Y_+^\lambda \subset \mathbb{Z}_{<0}^2$ and define $\mathcal{B}_-(\lambda)$ as the set of functions $Y_-^\lambda \rightarrow I$ satisfying (T1)–(T3) with Y_+^λ replaced by Y_-^λ .

We view Y_+^λ, Y_-^λ as Young diagrams in the southeast and northwest directions respectively. For example, take $\mathfrak{g} = \mathfrak{gl}(2|2)$ and $\lambda = 4\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + \epsilon_4 \in \mathcal{P}$:

$$Y_+^\lambda = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}, \quad Y_-^\lambda = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}$$

Definition 2.2. Let $i \in I_0$, $j \in I$ and $a \in \mathbb{C}^\times$. Define the ℓ -weights in \mathbf{R}_U :

- (i) $\boxed{j}_a := ((p_l(z))_{l \in I}; |\epsilon_j|)$ where $p_l(z) := (q_j \frac{1 - za\theta_j^{-1}q_j^{-1}}{1 - za\theta_j^{-1}q_j})^{\delta_{jl}}$ for $l \in I$;
- (ii) $A_{i,a} := \boxed{i}_{a\tau_i q^{-1}} \boxed{i+1}_{a\tau_i q^{-1}}^{-1}$, called *generalized simple root*;
- (iii) $\boxed{1}_a^* := \boxed{1}_{a\theta_1}^{-1}$ and $\boxed{i+1}_a^* := \boxed{i}_a^* A_{i,a\tau_i q^{-1}}$;
- (iv) $\boxed{\kappa}_a := \boxed{\kappa}_a^{-1}$ and $\boxed{i+1}_a := \boxed{i}_a A_{i,a\tau_i q^{-1}}$.

Call a the *spectral parameter* of the boxes \boxed{j}_a , \boxed{j}_a^* , \boxed{j}_a .

Example 2.3. If $\mathfrak{g} := \mathfrak{gl}(2|3)$, then $\tau_1 = q^{-1}$ and (compare with [26, §5.4.1])

$$\begin{aligned} \boxed{1}_a &\xrightarrow{A_{1,aq^2}^{-1}} \boxed{2}_a \xrightarrow{A_{2,aq^3}^{-1}} \boxed{3}_a \xrightarrow{A_{3,aq^2}^{-1}} \boxed{4}_a \xrightarrow{A_{4,aq}^{-1}} \boxed{5}_a, \\ \boxed{1}_a^* &\xrightarrow{A_{1,aq^{-2}}} \boxed{2}_a^* \xrightarrow{A_{2,aq^{-3}}} \boxed{3}_a^* \xrightarrow{A_{3,aq^{-2}}} \boxed{4}_a^* \xrightarrow{A_{4,aq^{-1}}} \boxed{5}_a^*. \end{aligned}$$

For $\lambda \in \mathcal{P}$, the $U_q(\mathfrak{g})$ -module $V_q(\lambda)$ is finite-dimensional [9, §3.3]; its dual space $V_q^*(\lambda) := \text{Hom}_{\mathbb{C}}(V_q(\lambda), \mathbb{C})$ is equipped with a $U_q(\mathfrak{g})$ -module structure:

$$\langle x\varphi, v \rangle := (-1)^{|\varphi||x|} \langle \varphi, \mathbb{S}(x)v \rangle \quad \text{for } x \in U_q(\mathfrak{g}), \varphi \in V_q^*(\lambda), v \in V_q(\lambda).$$

Theorem 2.4. Let $a \in \mathbb{C}^\times$ and $\lambda \in \mathcal{P}$. Let $V_q^\pm(\lambda; a)$, $V_q^{\pm*}(\lambda; a)$ be the pullbacks of the $U_q(\mathfrak{g})$ -modules $V_q(\lambda)$, $V_q^*(\lambda)$ by ev_a^\pm respectively. Then we have

$$(2.17) \quad \chi_q(V_q^+(\lambda; a)) = \sum_{T \in \mathcal{B}_-(\lambda)} \prod_{(i,j) \in Y_-^\lambda} \boxed{T(i,j)}_{aq^{2(j-i)+1}},$$

$$(2.18) \quad \chi_q(V_q^{+*}(\lambda; a)) = \sum_{T \in \mathcal{B}_-(\lambda)} \prod_{(i,j) \in Y_-^\lambda} \boxed{T(i,j)}_{aq^{2(i-j)+1}}^*,$$

$$(2.19) \quad \chi_q(V_q^-(\lambda; a)) = \sum_{T \in \mathcal{B}_+(\lambda)} \prod_{(i,j) \in Y_+^\lambda} \boxed{T(i,j)}_{aq^{2(j-i+M-N)+1}},$$

$$(2.20) \quad \chi_q(V_q^{-*}(\lambda; a)) = \sum_{T \in \mathcal{B}_+(\lambda)} \prod_{(i,j) \in Y_+^\lambda} \boxed{T(i,j)}_{aq^{2(i-j)+1}}.$$

In particular, $V_q^\pm(\lambda; a)$ and $V_q^{\pm*}(\lambda; a)$ have multiplicity free q -characters.

Remark 2.5. Applying $\varpi : \hat{\mathfrak{P}} \rightarrow \mathfrak{P}$ to Equation (2.19) recovers the character formula of $V_q(\lambda)$ in [9, Theorem 5.1]. The $U_{q^{-1}}(\hat{\mathfrak{g}})$ -module $h^*(V_q^-(\lambda; a))$ is $V(\lambda; a)$ in [51, Proposition 7.5], which proves Equation (2.19) by Remark 1.2. (One replaces \mathfrak{X}_{i,aq^m} in [51, Definition 6.2] with $\boxed{i}_{aq^{-m+2(M-N)}}$.)

For $i \in I$, let $U_q^{\geq i}(\hat{\mathfrak{g}})$ (resp. $U_q^{\geq i}(\mathfrak{g})$) be the subalgebra of $U_q(\hat{\mathfrak{g}})$ generated by the $s_{jk}^{(n)}, t_{jk}^{(n)}$ (resp. for $n = 0$) with $j, k \geq i$. Define

$$(2.21) \quad C_i(z) := \prod_{j \geq i} K_j^+(z\theta_j)^{(\epsilon_j, \epsilon_j)} \in Y_q(\mathfrak{g})[[z]].$$

The coefficients of $C_i(z)$ are central elements of $U_q^{\geq i}(\hat{\mathfrak{g}})$; see [51, Corollary 6.1].

Lemma 2.6. Let $i, l \in I$. The spectra of $C_i(z)$ on ℓ -weight spaces of ℓ -weights $\boxed{l}_a, \boxed{l}_a^*$ are $(q^{\frac{1-zaq^{-1}}{1-zaq}})^{\delta_{i \leq l}}$ and $(q^{-1} \frac{1-zaq}{1-zaq^{-1}})^{\delta_{i \leq l}}$ respectively, where $t_1 = \theta_1$ and $t_i = \tau_{i-1}^2 q^{-2}$ for $i > 1$. Moreover $\boxed{l}_a^* = \frac{(1-zaq^{-3})(1-zaq)}{(1-zaq^{-1})^2} \boxed{l}_a$.

Proof. The \boxed{l} -case is from Definition 2.2 (i). In particular the $A_{j,b}$ for $j \neq i-1$ do not contribute to the spectra of $C_i(z)$. The \boxed{l}^* -case is now clear from $\boxed{l}_a^* = \boxed{1}_{a\theta_1}^{-1} A_{1,a\tau_1 q^{-1}} A_{2,a\tau_2 q^{-1}} \cdots A_{l-1,a\tau_{l-1} q^{-1}}$. To compare \boxed{l}^* with \boxed{l} one may assume $l = \kappa$ by Definition 2.2 (iii)–(iv); the spectrum of $C_i(z)$ associated to the ℓ -weight $\boxed{\kappa}_a$ is $q^{-1} \frac{1-zaq}{1-zaq^{-1}}$, leading to the last identity. \square

We shall prove Equations (2.17)–(2.18); the idea is similar to [25, Lemma 4.7]. The proof of Equation (2.20) (and also (2.19)) is parallel and will be omitted.

Let S be $V_q^+(\lambda; a)$ or $V_q^{+*}(\lambda; a)$. If $\mu \in \mathbf{P}$ and $v \in S$ are such that $s_{ii}^{(0)} v = q^{(\mu, \epsilon_i)} v$ for all $i \in I$, then $|v| = |\mu|$. To compute the q -character of S , it is enough to determine the action of the $C_i(z)$ since it in turn implies the parity.

Let S_1 be an irreducible sub- $U_q^{\geq i}(\mathfrak{g})$ -module of S and $0 \neq v_1 \in S_1, \mu \in \mathbf{P}$ with

$$t_{jk}^{(0)} v_1 = 0, \quad s_{il}^{(0)} v_1 = q^{(\mu, \epsilon_l)} v_1 \quad \text{for } j, k, l \in I, \quad j > k.$$

Call μ the lowest weight of S_1 . By Schur Lemma and Gauss decomposition,

$$(2.22) \quad C_i(z)v = \prod_{j \geq i} \left(\frac{q^{(\mu, \epsilon_j)} - za\theta_j q^{-(\mu, \epsilon_j)}}{1 - za\theta_j} \right)^{(\epsilon_j, \epsilon_j)} v \quad \text{for } v \in S_1.$$

The strategy is to find all such triples (i, S_1, μ) .

Following Table (1.14) and Definition 2.1, define for \mathfrak{g}' the similar objects

$$\mathcal{P}' \subset \mathbf{P}, \quad Y_{\pm}'^{\lambda} \subset \mathbb{Z}^2, \quad \mathcal{B}_{\pm}'(\lambda), \quad V_q'(\lambda), \quad V_q'^*(\lambda)$$

with (M, N) replaced by (N, M) . The transpose of Young diagrams induces a bijection $\mathcal{P} \rightarrow \mathcal{P}', \lambda \mapsto \lambda^{\sharp}$ such that $(k, l) \in Y_+^{\lambda}$ if and only if $(l, k) \in Y_+^{\lambda^{\sharp}}$.

Lemma 2.7. *Let $\lambda \in \mathcal{P}$.*

- (1) *As $U_q(\mathfrak{g}')$ -modules $f^*(V_q(\lambda)) \cong V_q'^*(\lambda^{\sharp})$ and $f^*(V_q^*(\lambda)) \cong V_q'(\lambda^{\sharp})$.*
- (2) *If $T \in \mathcal{B}_-(\lambda)$, then $T'(k, l) := M + N + 1 - T(-l, -k)$ defines an element $T' \in \mathcal{B}_+^{\sharp}(\lambda^{\sharp})$. Moreover $T \mapsto T'$ is a bijection $\mathcal{B}_-(\lambda) \rightarrow \mathcal{B}_+^{\sharp}(\lambda^{\sharp})$.*

Proof. (2) is a lengthy but straightforward check by Definition 2.2. For (1), it suffices to establish the second isomorphism since f respects Hopf superalgebra structures. Let μ be the lowest weight of $V_q(\lambda)$ and define (see [51, §7.1])

$$r_i := \sharp\{j \in \mathbb{Z}_{>0} \mid (i, j) \in Y_+^{\lambda}\}, \quad c_j := \sharp\{i \in \mathbb{Z}_{>0} \mid (i, j) \in Y_+^{\lambda}\};$$

$$r'_i := \max(r_i - N, 0), \quad c'_j := \max(c_j - M, 0).$$

By Definition 2.1 and Remark 2.5 (see also Remark 2.8 with $i = 1$ below)

$$\lambda = \sum_{i=1}^M r_i \epsilon_i + \sum_{j=1}^N c'_j \epsilon_{M+j}, \quad \mu = \sum_{i=1}^M r'_{M+1-i} \epsilon_i + \sum_{j=1}^N c_{N+1-j} \epsilon_{M+j}.$$

If v is a lowest weight vector of $V_q(\lambda)$, then $V_q^*(\lambda)$ contains a highest weight vector v^* of weight $-\mu$, and $f^*(v^*) \in f^*(V_q^*(\lambda))$ is a highest weight vector of weight

$$c_1 \epsilon_1 + c_2 \epsilon_2 + \cdots + c_N \epsilon_N + r'_1 \epsilon_{N+1} + r'_2 \epsilon_{N+2} + \cdots + r'_M \epsilon_{M+N},$$

which is exactly λ^{\sharp} , leading to the desired isomorphism. \square

For $i \in I$ let $U_q^{\leq i}(\mathfrak{g}') := f^{-1}(U_q^{\geq \kappa+1-i}(\mathfrak{g}'))$; it is the subalgebra of $U_q(\mathfrak{g}')$ generated by the $s_{jk}^{(0)}, t_{jk}^{(0)}$ with $j, k \leq i$. To decompose $V_q(\lambda)$ (resp. $V_q^*(\lambda)$) with respect to lowest weights along the ascending chain of subalgebras of $U_q(\mathfrak{g})$

$$U_q^{\geq \kappa}(\mathfrak{g}) \subset U_q^{\geq \kappa-1}(\mathfrak{g}) \subset \cdots \subset U_q^{\geq 2}(\mathfrak{g}) \subset U_q^{\geq 1}(\mathfrak{g}) = U_q(\mathfrak{g})$$

is to decompose $V_q'(\lambda^{\sharp})$ with respect to highest (resp. lowest) weights along

$$U_q^{\leq 1}(\mathfrak{g}') \subset U_q^{\leq 2}(\mathfrak{g}') \subset \cdots \subset U_q^{\leq \kappa-1}(\mathfrak{g}') \subset U_q^{\leq \kappa}(\mathfrak{g}') = U_q(\mathfrak{g}'),$$

Remark 2.8. The decompositions were described in [51, §7.1] in terms of $\mathcal{B}_+^{\sharp}(\lambda^{\sharp})$, and equivalently of $\mathcal{B}_-(\lambda)$ by Lemma 2.7 (2).

- (1) $V_q(\lambda)$ admits a basis $(v_T : T \in \mathcal{B}_-(\lambda))$ such that v_T is contained in an irreducible sub- $U_q^{\geq i}(\mathfrak{g})$ -module of lowest weight $\mu_T^{\geq i}$ for $i \in I$.
- (2) $V_q^*(\lambda)$ admits a basis $(w_T : T \in \mathcal{B}_-(\lambda))$ such that w_T is contained in an irreducible sub- $U_q^{\geq i}(\mathfrak{g})$ -module of lowest weight $-\nu_T^{\geq i}$ for $i \in I$.

$\mu_T^{\geq i}$ and $\nu_T^{\geq i}$ are defined as follows. Set $Y_T^{\geq i} := \{(k, l) \in Y_-^\lambda \mid T(k, l) \geq i\}$ and

$$r_k := \#\{l \in \mathbb{Z} \mid (-k, -l) \in Y_T^{\geq i}\}, \quad c_l := \#\{k \in \mathbb{Z} \mid (-k, -l) \in Y_T^{\geq i}\}.$$

If $i > M$, then $\begin{cases} \mu_T^{\geq i} = c_1 \epsilon_{M+N} + c_2 \epsilon_{M+N-1} + \cdots + c_{M+N+1-i} \epsilon_i, \\ \nu_T^{\geq i} = c_1 \epsilon_i + c_2 \epsilon_{i+1} + \cdots + c_{M+N+1-i} \epsilon_{M+N}. \end{cases}$ If $i \leq M$, then

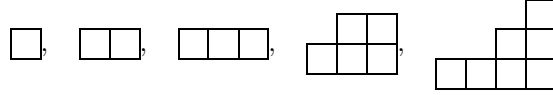
$$\begin{aligned} \mu_T^{\geq i} &= c_1 \epsilon_{M+N} + c_2 \epsilon_{M+N-1} + \cdots + c_N \epsilon_{M+1} + r'_1 \epsilon_M + r'_2 \epsilon_{M-1} + \cdots + r'_{M+1-i} \epsilon_i, \\ \nu_T^{\geq i} &= r_1 \epsilon_i + r_2 \epsilon_{i+1} + \cdots + r_{M+1-i} \epsilon_M + c'_1 \epsilon_{M+1} + c'_2 \epsilon_{M+2} + \cdots + c'_N \epsilon_{M+N}, \end{aligned}$$

where $r'_k := \max(r_k - N, 0)$ and $c'_l := \max(c_l - M + i - 1, 0)$.

Example 2.9. To illustrate Lemma 2.7 (2) and Remark 2.8, let $\mathfrak{g} = \mathfrak{gl}(2|3)$ and $\lambda = 4\epsilon_1 + 2\epsilon_2 + \epsilon_3 \in \mathcal{P}$. We represent elements in $\mathcal{B}_-(\lambda)$ and $\mathcal{B}'_+(\lambda^\sharp)$ by Young tableaux of shapes λ, λ^\sharp respectively. Let $T \in \mathcal{B}_-(\lambda)$ be such that

$$\mathcal{B}_-(4\epsilon_1 + 2\epsilon_2 + \epsilon_3) \ni T = \begin{array}{|c|c|c|c|} \hline & & 1 & \\ \hline & 2 & 2 & \\ \hline 1 & 3 & 4 & 5 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array} = T' \in \mathcal{B}'_+(3\epsilon_1 + 2\epsilon_2 + \epsilon_3 + \epsilon_4).$$

The Young diagrams $Y_T^{\geq i}$ with descending order on $5 \geq i \geq 1$ become:



Correspondingly, the pairs $(\mu_T^{\geq i}, \nu_T^{\geq i})$ from $i = 5$ to $i = 1$ are:

$$\begin{aligned} &(\epsilon_5, \epsilon_5), \quad (\epsilon_4 + \epsilon_5, \epsilon_4 + \epsilon_5), \quad (\epsilon_3 + \epsilon_4 + \epsilon_5, \epsilon_3 + \epsilon_4 + \epsilon_5), \\ &(\epsilon_3 + 2\epsilon_4 + 2\epsilon_5, 3\epsilon_2 + \epsilon_3 + \epsilon_4), \quad (\epsilon_2 + \epsilon_3 + 2\epsilon_4 + 3\epsilon_5, 4\epsilon_1 + 2\epsilon_2 + \epsilon_3). \end{aligned}$$

Proof of Equations (2.17)–(2.18). Let us define $g_i(z), g_i^*(z) \in \mathbb{C}[[z]]^\times$ for $i \in I$:

$$g_i(z) := \prod_{(k,l) \in Y_T^{\geq i}} q \frac{1 - zaq^{2(l-k)}}{1 - zaq^{2(l-k+1)}}, \quad g_i^*(z) := \prod_{(k,l) \in Y_T^{\geq i}} \left(q^{-1} \frac{1 - zat_i q^{2(k-l+1)}}{1 - zat_i q^{2(k-l)}} \right).$$

By Lemma 2.6, it suffices to prove that: for $i \in I$,

$$C_i(z)v_T = g_i(z)v_T \quad \text{in } V_q^+(\lambda; a), \quad C_i(z)w_T = g_i^*(z)w_T \quad \text{in } V_q^{+*}(\lambda; a).$$

This is divided into two cases: $i > M$ or $i \leq M$.

Assume $i > M$. Then $T(-k, -l) \geq i$ if and only if $1 \leq l \leq M + N - i + 1$ and $1 \leq k \leq c_l$. It follows from Equation (1.8) that

$$\begin{aligned} g_i(z) &= \prod_{l=1}^{M+N-i+1} \prod_{k=1}^{c_l} q \frac{1 - zaq^{2(k-l)}}{1 - zaq^{2(k-l+1)}} = \prod_{l=1}^{M+N-i+1} \frac{1 - zaq^{2(1-l)}}{q^{-c_l} - zaq^{2(1-l)+c_l}} \\ &= \prod_{j=i}^{M+N} \left(\frac{q^{(\mu_T^{\geq i}, \epsilon_j)} - za\theta_j q^{-(\mu_T^{\geq i}, \epsilon_j)}}{1 - za\theta_j} \right)^{(\epsilon_j, \epsilon_j)}, \\ g_i^*(z) &= \prod_{l=1}^{M+N-i+1} \prod_{k=1}^{c_l} q^{-1} \frac{1 - zat_i q^{2(l-k+1)}}{1 - zat_i q^{2(l-k)}} = \prod_{l=1}^{M+N-i+1} \frac{1 - zat_i q^{2l}}{q^{c_l} - zat_i q^{2l-c_l}} \\ &= \prod_{j=i}^{M+N} \left(\frac{q^{-(\nu_T^{\geq i}, \epsilon_i)} - za\theta_j q^{(\mu_T^{\geq i}, \epsilon_i)}}{1 - za\theta_j} \right)^{(\epsilon_j, \epsilon_j)}. \end{aligned}$$

Here in the last equation we used $t_i q^{2l} = \tau_{i-1}^2 q^{2l-2} = \theta_i q^{2l-2} = \theta_{i+l-1}$.

Assume $i \leq M$. Then $T(-k, -l) \geq i$ if and only if $(1 \leq l \leq N, 1 \leq k \leq c_l)$ or $(1 \leq k \leq M+1-i, N+1 \leq l \leq N+r'_k)$. This gives

$$\begin{aligned} g_i(z) &= \left(\prod_{l=1}^N \prod_{k=1}^{c_l} q \frac{1 - zaq^{2(k-l)}}{1 - zaq^{2(k-l+1)}} \right) \times \left(\prod_{k=1}^{M+1-i} \prod_{l=1}^{r'_k} q \frac{1 - zaq^{2(k-l-N)}}{1 - zaq^{2(k-l-N+1)}} \right) \\ &= \prod_{j=i}^{M+N} \left(\frac{q^{(\mu_T^{\geq i, \epsilon_j})} - za\theta_j q^{-(\mu_T^{\geq i, \epsilon_j})}}{1 - za\theta_j} \right)^{(\epsilon_j, \epsilon_j)}. \end{aligned}$$

Notice that $T(-k, -l) \geq i$ if and only if $(1 \leq k \leq M+1-i, 1 \leq l \leq r_k)$ or $(1 \leq l \leq N, M-i+2 \leq k \leq M-i+1+c'_l)$. This gives

$$\begin{aligned} g_i^*(z) &= \left(\prod_{k=1}^{M+1-i} \prod_{l=1}^{r_k} q^{-1} \frac{1 - zat_i q^{2(l-k+1)}}{1 - zat_i q^{2(l-k)}} \right) \left(\prod_{l=1}^N \prod_{k=1}^{c'_l} q^{-1} \frac{1 - zat_i q^{2(l-k-M+i)}}{1 - zat_i q^{2(l-k-M+i-1)}} \right) \\ &= \left(\prod_{k=1}^{M+1-i} \frac{q^{-r_k} - zat_i q^{2(1-k+r_k)}}{1 - zat_i q^{2(1-k)}} \right) \left(\prod_{l=1}^N \frac{1 - zat_i q^{2(l-1-M+i)}}{q^{c'_l} - zat_i q^{2(l-1-M+i-c'_l)}} \right) \\ &= \prod_{j=i}^{M+N} \left(\frac{q^{-(\nu_T^{\geq i, \epsilon_i})} - za\theta_j q^{(\mu_T^{\geq i, \epsilon_i})}}{1 - za\theta_j} \right)^{(\epsilon_j, \epsilon_j)}. \end{aligned}$$

The last identity comes from $t_i q^{2(l-1-M+i)} = \theta_{M+l}$ and $t_i q^{2(1-k)} = \theta_{i+k-1}$.

In both cases, $g_i(z)$ and $g_i^*(z)$ become Equation (2.22) with $\mu = \mu_T^{\geq i}$ and $-\nu_T^{\geq i}$ respectively, and this completes the proof of Equations (2.17)–(2.18). \square

Let $\widehat{\mathcal{Q}}^-$ be the submonoid of \mathbf{R} generated by the $A_{i,a}^{-1}$ with $i \in I_0$ and $a \in \mathbb{C}^\times$.

Corollary 2.10. *Let $i \in I_0$, $a \in \mathbb{C}^\times$ and $m \in \mathbb{C}^\times$. We have*

$$(2.23) \quad W_{m,a}^{(i)} \cong V_q^+(m\varpi_i; aq^{M-N-i}) \cong V_q^-(m\varpi_i; aq^{N-M+i-2m}) \text{ if } i \leq M,$$

$$(2.24) \quad W_{m,a}^{(i)} \cong V_q^{-*}(\lambda_m^{(i)}; aq^{M+N-2-i}) \simeq V_q^{+*}(\lambda_m^{(i)}; aq^{i-M-N+2m-2}) \text{ otherwise.}$$

Here for $i > M$, the Young diagram of $\lambda_m^{(i)} \in \mathcal{P}$ is a rectangle with m rows and $\kappa - i$ columns. An ℓ -weight of $W_{m,a}^{(i)}$ different from $\varpi_{m,a}^{(i)}$ must belong to $\varpi_{m,a}^{(i)} A_{i,aq_i}^{-1} \widehat{\mathcal{Q}}^-$.

Proof. Assume $i \leq M$. The Young diagram $Y_-^{m\varpi_i}$ is a rectangle with i rows and m columns. Let $H \in \mathcal{B}_-(m\varpi_i)$ be such that $H(-k, -l) = i+1-k$ for $1 \leq k \leq i$. Then $v_H \in V_q^+(m\varpi_i; a\tau_i q^{-1})$ in Remark 2.8 is a highest ℓ -weight vector of ℓ -weight

$$m_H = \prod_{l=1}^m \prod_{k=1}^i \boxed{k}_{a\tau_i q^{2(i+1-k-l)}} = \prod_{l=1}^m Y_{i,aq^{2-2l}} = \varpi_{m,a}^{(i)}.$$

Here we used $\prod_{k=1}^i \boxed{k}_{a\tau_i q^{2(i+1-k-l)}} = Y_{i,aq^{2-2l}}$ and $\theta_i = \tau_i^2 = q^{2(M-N+1-i)}$ for $1 \leq i \leq M$, based on Definitions 1.7 and 2.2. This proves the first isomorphism of (2.23); the second one is a consequence of Equations (2.17) and (2.19). If $T \in \mathcal{B}_-(m\varpi_i)$ and $T \neq H$, then $T(-k, -l) \geq i+1-k$ and $T(-1, -1) > i$. The ℓ -weight property of $W_{m,a}^{(i)}$ follows from Definition 2.2 (ii) and Equation (2.17):

$$m_T m_H^{-1} \in \boxed{i+1}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} \widehat{\mathcal{Q}}^-.$$

Assume $i > M$. Let v be the highest ℓ -weight of $V_q^{-*}(\lambda_m^{(i)}; b)$. By Equation (1.6),

$$K_p^+(z)v = v \quad \text{for } p \leq i, \quad K_p^+(z)v = \frac{1 - zb}{q^{-m} - zbq^m}v \quad \text{for } p > i.$$

v is of ℓ -weight $\varpi_{m,brjq}^{(j)}$, proving the first isomorphism of (2.24). Since $\boxed{l}_a^* \equiv \boxed{l}'_a$ for $l \in I$, the second isomorphism of (2.24) is deduced from Equations (2.18) and (2.20). let $H \in \mathcal{B}_+(\lambda_m^{(i)})$ be such that $H(k, l) = i + l$ for $1 \leq l \leq M + N - i$. The monomial m'_H associated to H in Equation (2.20) is the highest ℓ -weight. If $T \in \mathcal{B}_+(\lambda_m^{(i)})$ and $T \neq H$, then $T(k, l) \leq i + l$ and $T(1, 1) \leq i$. By Definition 2.2 (iv) and Equation (2.20):

$$m'_T m'^{-1}_H \in \boxed{i}'_{a\tau_i^{-1}} \boxed{i+1}'_{a\tau_i^{-1}} \hat{\mathcal{Q}}^- = A_{i,aq^{-1}}^{-1} \hat{\mathcal{Q}}^-,$$

proving the ℓ -weight property of $W_{m,a}^{(i)}$. \square

The ℓ -weight property is similar to [29, Lemma 4.4]; $W_{m,a}^{(i)}$ in [29] is $W_{m,aq_i^{2m-2}}^{(i)}$ here. Let $\varpi_{m,a}^{(M-)} := \prod_{l=1}^m Y_{M,aq^{2l-2}}^{-1}$ and $W_{m,a}^{(M-)} := L(\varpi_{m,a}^{(M-)})$. Similarly we have

$$(2.25) \quad W_{m,a}^{(M-)} \simeq V_q^{-*}(\lambda_m; aq^{N-2}) \simeq V_q^{+*}(\lambda_m; aq^{2m-2-N}).$$

where $\lambda_m \in \mathcal{P}$ is such that its Young diagram is a rectangle with m rows and N columns. If $\varpi_{m,a}^{(M-)} \mathbf{n} \in \text{wt}_\ell(W_{m,a}^{(M-)})$ and $\mathbf{n} \neq 1$, then $\mathbf{n} \in A_{M,aq^{-1}}^{-1} \hat{\mathcal{Q}}^-$.

For $i, j \in I_0$ write $i \sim j$ if $j = i \pm 1$. Recall $q_{ij} = q^{(\alpha_i, \alpha_j)}$.

Lemma 2.11. *Let $i \in I_0 \setminus \{M, M+1\}$ and $a \in \mathbb{C}^\times$. In \mathbf{R} we have*

$$A_{i,a} \equiv Y_{i,aq} Y_{i,aq^{-1}} \prod_{j \in I_0: j \sim i} Y_{j,a}^{-1}, \quad A_{M,a} \equiv \prod_{j \in I_0: j \sim M} Y_{j,a}^{-1}.$$

If $N > 2$, then $A_{M+1,a} \equiv Y_{M+1,aq} Y_{M+1,aq^{-1}} Y_{M,a} Y_{M+2,a}^{-1}$. If $N = 2$, then $A_{M+1,a} \equiv Y_{M+1,aq} Y_{M+1,aq^{-1}} Y_{M,a}$. In terms of the Ψ , for $i \in I_0$ we have:

$$(2.26) \quad A_{i,a} \equiv \frac{\Psi_{i,aq_i^{-2}}}{\Psi_{i,a\hat{q}_i^2}} \prod_{j \in I_0: j \sim i} \frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}}},$$

where $\hat{q}_i = q_i$ for $i \neq M$ and $\hat{q}_M = q^{-1}$.

Proof. Straightforward by Definitions 1.7 and 2.2. \square

3. LENGTH-TWO REPRESENTATIONS

We describe certain length-two representations in category \mathcal{O} .

Definition 3.1. Let $m \in \mathbb{Z}_{>0}$, $s \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{C}^\times$ and $i \in I_0$. Define ℓ -weights:

- (i) $\mathbf{n}_{i,a}^+ := \Psi_{i,a}^{-1} \Psi_{i,aq_i^{-2}} \prod_{j \in I_0: j \sim i} \Psi_{j,aq_{ij}^{-1}}$, $\mathbf{n}_{i,a}^- := \Psi_{i,a} \Psi_{i,a\hat{q}_i^2}^{-1} \prod_{j \in I_0: j \sim i} \Psi_{j,aq_{ij}}^{-1}$;
- (ii) if $i \neq M$, then $\mathbf{d}_{m,a}^{(i,s)} := \varpi_{m,aq_i^{2m+1}}^{(i)} \varpi_{m+s,aq_i^{2m-1}}^{(i)} \prod_{l=1}^m A_{i,aq_i^{2l}}^{-1} \in \mathbf{R}_U$;
- (iii) $\mathbf{d}_{m,a}^{(M,s)} := \varpi_{s,aq^{-1}}^{(M)} \prod_{j \in I_0: j \sim M} \varpi_{m,aq_j^{2m}}^{(j)} \in \mathbf{R}_U$.

$N_{i,a}^\pm := L(\mathbf{n}_{i,a}^\pm)$ are $Y_q(\mathfrak{g})$ -modules and $D_{m,a}^{(i,s)} := L(\mathbf{d}_{m,a}^{(i,s)})$ are $U_q(\hat{\mathfrak{g}})$ -modules.

Remark 3.2. Let us rewrite $\mathbf{d}_{m,a}^{(i,s)}$ in terms of the Ψ using Equation (2.26):

$$\mathbf{d}_{m,a}^{(i,s)} \equiv \frac{\Psi_{i,aq_i^{-2s}}}{\Psi_{i,a}} \prod_{j \in I_0: j \sim i} \frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}^{-2m-1}}}.$$

In the non-graded case $N = 0$, we can identify $\mathbf{n}_{i,a}^+$ with Ψ in [33, (6.13)] and $\mathbf{m}_{i,a}^{(2)}$ in [18, (6.2)], $\mathbf{d}_{m,a}^{(i,s)}$ with $\tilde{\Psi}_i^{(-s, 2m-1)}$ in [24, §4.3]. Notice that $\mathbf{d}_{m,a}^{(i,s)}$ satisfies the condition of “minimal affinization by parts” in [13, Theorem 2].

Theorem 3.3. *Let $i \in I_0$ and $a \in \mathbb{C}^\times$. The $Y_q(\mathfrak{g})$ -module $N_{i,a}^+ \otimes L_{i,a}^+$ has a Jordan–Hölder series of length two and in the Grothendieck ring $K_0(\mathcal{O})$:*

$$(3.27) \quad [N_{i,a}^+ \otimes L_{i,a}^+] = [L_{i,aq_i^{-2}}^+] \prod_{j \in I_0: j \sim i} [L_{j,aq_{ij}^{-1}}^+] + [D][L_{i,aq_i^2}^+] \prod_{j \in I_0: j \sim i} [L_{j,aq_{ij}}^+].$$

Here $D = L(\mathbf{n}_{i,a}^+ \Psi_{i,a} \Psi_{i,aq_i^2}^{-1} A_{i,a}^{-1} \prod_{j \sim i} \Psi_{j,aq_{ij}}^{-1})$ is one-dimensional.

When $i = M$, the two monomials at the right-hand side of Equation (3.27) has a common factor $[L_{M,aq^{-2}}^+]$. This is a special feature of quantum affine superalgebras.

Theorem 3.4. *Let $i \in I_0 \setminus \{M\}$, $a \in \mathbb{C}^\times$ and $m, s \in \mathbb{Z}_{>0}$. There are short exact sequences of $U_q(\widehat{\mathfrak{g}})$ -modules whose first and third terms are irreducible:*

$$\begin{aligned} D_{m,a}^{(i,s)} &\hookrightarrow W_{m,aq_i^{2m+1}}^{(i)} \otimes W_{m+s,aq_i^{2m-1}}^{(i)} \twoheadrightarrow W_{m+s+1,aq_i^{2m+1}}^{(i)} \otimes W_{m-1,aq_i^{2m-1}}^{(i)}, \\ D_{m+s,aq_i^{-2s}}^{(i,0)} \otimes W_{m,aq_i^{2m+1}}^{(i)} &\hookrightarrow W_{m+s,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s)} \twoheadrightarrow W_{m+s+1,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s-1)}. \end{aligned}$$

The assumption $i \neq M$ is necessary because $\dim W_{m,a}^{(M)} = 2^{MN}$ for $m \geq N$. Equation (3.27) corresponds to [33, (6.14)] and [18, Proposition 6.8], and can be thought of as a two-term Baxter TQ relation for $Y_q(\mathfrak{g})$. The exact sequences of Theorem 3.4 are extended T-systems [40, 29], the initial case $s = 0$ being the T-system in [42]; see Theorem 8.3.

The proof of Theorem 3.3, given in Section 4, is similar to [33, (6.14)], based on q -characters. Theorem 3.4 is more involved and requires cyclicity of tensor products of KR modules; its proof is postponed to Section 8.

We make crucial use of the idea that $D_{m,a}^{(i,s)}$ admits an injective resolution by tensor products of KR modules of the same Dynkin node for $i \neq M$.

Lemma 3.5. *Let $m \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$ and $i \in I_0 \setminus \{M\}$. If $\varpi_{m,a}^{(i)} \mathbf{n} \in \text{wt}_\ell(W_{m,a}^{(i)})$ and $\mathbf{n} \neq 1$, then either $\mathbf{n} = A_{i,aq_i}^{-1} A_{i,aq_i^{-1}}^{-1} \cdots A_{i,aq_i^{3-2l}}^{-1}$ for some $1 \leq l \leq m$, or \mathbf{n} belongs to $A_{i,aq_i}^{-1} A_{j,aq_i^2}^{-1} \widehat{\mathcal{Q}}^-$ where $j \in I_0$ and $j \sim i$.*

Proof. We only consider the case $i < M$; the other case is similar. Let us be in the situation of the proof of Corollary 2.10. By Equation (2.17), $\mathbf{n} = m_T m_H^{-1}$ for a unique $T \in \mathcal{B}_-(m\varpi_i)$ with $T(-1, -1) > i$ and $T(-k, -l) \geq i + 1 - k$. If $T(-1, -1) > i + 1$, then using $\tau_{i+1} = q^{-1}\tau_i$ we obtain

$$m_T m_H^{-1} \in \boxed{i+2}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} A_{i+1,aq^2}^{-1} \widehat{\mathcal{Q}}^-.$$

If $T(-2, -1) > i - 1$, then together with $T(-1, -1) > i$ we have

$$m_T m_H^{-1} \in \boxed{i+1}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \boxed{i}_{a\tau_i q^2} \boxed{i-1}_{a\tau_i q^2}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} A_{i-1,aq^2}^{-1} \widehat{\mathcal{Q}}^-.$$

Suppose $T(-1, -1) = i + 1$ and $T(-2, -1) = i - 1$. There exists $1 \leq l \leq m$ such that the only difference between T, H is at $(-1, -j)$ with $1 \leq j \leq l$, and

$$m_T m_H^{-1} = \prod_{j=1}^l \boxed{i+1}_{a\tau_i q^{2-2j}} \boxed{i}_{a\tau_i q^{2-2j}}^{-1} = \prod_{j=1}^l A_{i,aq^{3-2j}}^{-1}.$$

This completes the proof of the lemma. \square

Corollary 3.6. *Let $m, s \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$ and $i \in I_0 \setminus \{M\}$.*

- (1) *For $1 \leq l \leq s$, we have $\mathbf{d}_{m,a}^{(i,s)} A_{i,aq_i^{-2}}^{-1} \cdots A_{i,aq_i^{3-2l}}^{-1} \in \text{wt}_\ell(D_{m,a}^{(i,s)})$ and its associated ℓ -weight space is one-dimensional.*
- (2) *If $\mathbf{d}_{m,a}^{(i,s)} \mathbf{n} \in \text{wt}_\ell(D_{m,a}^{(i,s)})$ is not of the form of (1) and $\mathbf{n} \neq 1$, then $\mathbf{n} \in \{A_{j,aq_i}^{-1} A_{i,aq_i^{2m+2}}^{-1} \mid j \in I_0, j \sim i\} \widehat{\mathcal{Q}}^-$.*

Proof. Set $T := W_{m, aq_i^{2m+1}}^{(i)} \otimes W_{m+s, aq_i^{2m-1}}^{(i)}$ and $S := L(\varpi_{m, aq_i^{2m+1}}^{(i)} \varpi_{m+s, aq_i^{2m-1}}^{(i)})$. Then S is a sub-quotient of T . Let $\lambda := (2m+s)\varpi_i$. By Corollary 2.10,

$$(A) \dim T_{q^{\lambda-k\alpha_i}} = \min(m+1, k+1) \text{ for } 0 \leq k \leq m+s.$$

1. Let $v_0 \in S$ be a highest ℓ -weight vector, U_i be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $(x_{i,n}^\pm, \phi_{i,n}^\pm)_{n \in \mathbb{Z}}$, and $S^i := U_i v_0 \subseteq S$. Since U_i is a quotient algebra of $U_{q_i}(\widehat{\mathfrak{sl}}_2)$, we can view S^i as a $U_{q_i}(\widehat{\mathfrak{sl}}_2)$ -module of highest ℓ -weight [26, §2]

$$\mathbf{m}_i := (Y_{aq_i^{2m+1}} Y_{aq_i^{2m-1}} \cdots Y_{aq_i^3})(Y_{aq_i^{2m-1}} Y_{aq_i^{2m-3}} \cdots Y_{aq_i^{1-2s}}).$$

S^i is spanned by the $x_{i,n_1}^- x_{i,n_2}^- \cdots x_{i,n_k}^- v_0$. If $w \in S^i$ is annihilated by the $x_{i,n}^+$, then $x_{j,n}^+ w = 0 \in S$ for all $j \in I_0 \setminus \{i\}$ (because $[x_{j,n}^+, x_{i,k}^-] = 0$) and $w \in \mathbb{C}v_0$. The $U_{q_i}(\widehat{\mathfrak{sl}}_2)$ -module S^i is irreducible and has a factorization [14, Theorem 4.8]:

$$S^i \cong L^i(Y_{aq_i^{2m+1}} Y_{aq_i^{2m-1}} \cdots Y_{aq_i^{1-2s}}) \otimes L^i(Y_{aq_i^{2m-1}} Y_{aq_i^{2m-3}} \cdots Y_{aq_i^3}),$$

where $L^i(\mathbf{n})$ denotes the irreducible $U_{q_i}(\widehat{\mathfrak{sl}}_2)$ -module of highest ℓ -weight \mathbf{n} (for \mathbf{n} a product of the Y_b). For $k \in \mathbb{Z}_{>0}$, let $V_k \subseteq S^i$ be the subspace spanned by the $x_{i,n_1}^- x_{i,n_2}^- \cdots x_{i,n_k}^- v_0$ with $n_l \in \mathbb{Z}$ for $1 \leq l \leq k$. Then $V_k = S_{q^{\lambda-k\alpha_i}}$. Based on the q -character of S^i with respect to the spectra of $\phi_i^+(z)$ in [26, §4.1], for $-1 \leq l < s$:

$$(B) \dim S_{q^{\lambda-k\alpha_i}} = \min(m, k+1) \text{ for } 1 \leq k \leq m+s;$$

$$(C) \mathbf{m}_i \prod_{t=-l}^m (Y_{aq_i^{2t+1}}^{-1} Y_{aq_i^{2t-1}}^{-1}) \text{ is not an } \ell\text{-weight of the } U_{q_i}(\widehat{\mathfrak{sl}}_2)\text{-module } S^i.$$

2. By (A)–(B), $\{\mathbf{n} \in \text{wt}_\ell(T) \setminus \text{wt}_\ell(S) \mid \varpi(\mathbf{n}) = \lambda - (m+l)\alpha_i\} = \{\mathbf{n}_l\}$ for $0 \leq l \leq s$, the multiplicity of \mathbf{n}_l in $\chi_q(T) - \chi_q(S)$ is one, and $L(\mathbf{n}_0)$ is a sub-quotient of T . Comparing the spectra of $\phi_i^+(z)$ by (C) and Lemma 3.5, we obtain: $\mathbf{n}_0 = \mathbf{d}_{m,a}^{(i,s)}$ and $\mathbf{n}_l = \mathbf{d}_{m,a}^{(i,s)} A_{i,a}^{-1} A_{i,aq_i^{-2}}^{-1} \cdots A_{i,aq_i^{2-2l}}^{-1}$. Part (2) follows by viewing $D_{m,a}^{(i,s)}$ as a sub-quotient of T . If $(D_{m,a}^{(i,s)})_{q^{\lambda-(m+l)\alpha_i}} \neq 0$ for $1 \leq l \leq s$, then necessarily $\mathbf{n}_l \in \text{wt}_\ell(D_{m,a}^{(i,s)})$ and its ℓ -weight space is one-dimensional, proving (1).

3. Let $w_0 \in D_{m,a}^{(i,s)}$ be a highest ℓ -weight vector. Then $x_{i,0}^+ w_0 = 0$ and $\phi_{i,0}^+ w_0 = q_i^s w_0$. Since the triple $(x_{i,0}^+, x_{i,0}^-, \phi_{i,0}^+)$ generates a quotient algebra of $U_{q_i}(\widehat{\mathfrak{sl}}_2)$, we have $(D_{m,a}^{(i,s)})_{q^{\lambda-(m+l)\alpha_i}} \ni (x_{i,0}^-)^l w_0 \neq 0$ for $1 \leq l \leq s$. \square

Corollary 3.6 corresponds to the first formula of [24, Lemma 4.8] whose proof utilized a delicate elimination theorem of ℓ -weights [31, Theorem 5.1]. Our arguments seem to work in *loc. cit.*, by realizing $L(\widetilde{\Psi}_i^{(N,M)})$ as sub-quotients of tensor products of KR modules. The Dynkin node $i = M$ is studied separately.

Corollary 3.7. *Let $m, s \in \mathbb{Z}_{>0}$ and $a \in \mathbb{C}^\times$.*

- (1) $\mathbf{d}_{m,a}^{(M,s)} A_{M,a}^{-1} \in \text{wt}_\ell(D_{m,a}^{(M,s)})$ and the ℓ -weight space is one-dimensional.
- (2) $(\mathbf{d}_{m,a}^{(M,s)})^{-1} \text{wt}_\ell(D_{m,a}^{(M,s)}) \subset \left(\{A_{j,aq_j^{2m+1}}^{-1} \mid j \in I_0, j \sim M\} \widehat{\mathcal{Q}}^- \right) \cup \{1, A_{M,a}^{-1}\}.$

Proof. Assume $M, N > 1$ without loss of generality. Let $\mathbf{n} \in (\mathbf{d}_{m,a}^{(M,s)})^{-1} \text{wt}_\ell(D_{m,a}^{(M,s)})$ with $\mathbf{n} \notin \{A_{M+1,aq^{2m-1}}^{-1}, A_{M-1,aq^{2m+1}}^{-1}\} \widehat{\mathcal{Q}}^-$ and $\mathbf{n} \neq 1$.

Firstly, set $\lambda := s\varpi_M + m\varpi_{M-1}$. Then $\lambda \in \mathcal{P}$ and its Young diagram Y_+^λ is formed of (k, l) where either $(1 \leq k < M, 1 \leq l \leq s+m)$ or $(k = M, 1 \leq l \leq s)$. Consider the evaluation module $S := V_q^-(\lambda; aq^{N-2s-1})$. Let $H \in \mathcal{B}_+(\lambda)$ be such that $H(k, l) = k$. The monomial m_H attached to H in Equation (2.19) is the highest ℓ -weight of S . From the proof of Corollary 2.10 we see that

$$m_H = (Y_{M,aq^{1-2s}} \cdots Y_{M,aq^{-3}} Y_{M,aq^{-1}})(Y_{M-1,aq^2} Y_{M-1,aq^4} \cdots Y_{M-1,aq^{2m}}).$$

In particular, the spectral parameters at the boxes (M, s) and $(M-1, s+m)$ of H are $a\tau_M q^{-1}$ and $a\tau_{M-1} q^{2m}$ respectively. Let $T \in \mathcal{B}_+(\lambda)$ and $T \neq H$. If $T(M-1, s+m) \geq M$, then by Definition 2.2 (ii) and Equation (2.19),

$$m_T m_H^{-1} \in \boxed{M}_{a\tau_{M-1} q^{2m}} \boxed{M-1}_{a\tau_{M-1} q^{2m}}^{-1} \widehat{\mathcal{Q}}^- = A_{M-1, aq^{2m+1}}^{-1} \widehat{\mathcal{Q}}^-.$$

If $T(M-1, s+m) < M$, then $T(k, l) = k$ for $k < M$ and by Equation (2.19):

- (i) the ℓ -weight space S_{m_T} is also the one-dimensional weight space $S_{\varpi(m_T)}$;
- (ii) $m_T m_H^{-1} A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \geq M$;
- (iii) if $m_T m_H^{-1} A_{M,a}$ is a product of the $A_{M,b}^{-1}$, then $m_T m_H^{-1} A_{M,a} = 1$.

Here we used Definition 2.1 (T3) and $T(M, l) \geq M$, $T(M, s) > M$.

Secondly, viewing $D_{m,a}^{(M,s)}$ as a sub-quotient of $S \otimes W_{m, aq^{-2m}}^{(M+1)}$ gives $\mathbf{n} = \mathbf{n}_1 \mathbf{n}_2$ with $m_H \mathbf{n}_1 \in \text{wt}_\ell(S)$ and $\mathbf{n}_2 \varpi_{m, aq^{-2m}}^{(M+1)} \in \text{wt}_\ell(W_{m, aq^{-2m}}^{(M+1)})$. Since $\mathbf{n} \notin A_{M+1, aq^{-2m-1}}^{-1} \widehat{\mathcal{Q}}^-$, by Corollary 2.10, $\mathbf{n}_2 = 1$ and $m_H \mathbf{n} \in \text{wt}_\ell(S)$. Since $\mathbf{n} \notin A_{M-1, aq^{2m+1}}^{-1} \widehat{\mathcal{Q}}^-$, (ii)–(iii) hold by replacing $m_T m_H^{-1}$ with \mathbf{n} , and $\dim(D_{m,a}^{(M,s)})_{\mathbf{d}_{m,a}^{(M,s)} \mathbf{n}} = 1$.

Thirdly, for $t \in \mathbb{Z}_{>0}$, let $\mu_t \in \mathcal{P}$ be such that its Young diagram $Y_{-}^{\mu_t}$ is formed of $(-k, -l)$ where either $(1 \leq l < N, 1 \leq k \leq m+t)$ or $(l = N, 1 \leq k \leq t)$. Consider the evaluation module $S_t := V_q^{+*}(\mu_t; aq^{2t-1-N})$. Let $H_t \in \mathcal{B}_-(\mu_t)$ be such that $H_t(-k, -l) = M+N+1-l$. The monomial $m_{H_t}^*$ in Equation (2.18) is the highest ℓ -weight of S_t and by Corollary 2.10 and Equation (2.25):

$$m_{H_t}^* \equiv \varpi_{m, aq^{-2m}}^{(M+1)} \varpi_{t, aq}^{(M-)}.$$

The spectral parameters at the boxes $(-t, -N)$ and $(-t-m, 1-N)$ of H_t are $a\tau_M^{-1} q$ and $a\tau_{M+1}^{-1} q^{-2m}$ respectively. Let $T \in \mathcal{B}_-(\mu_t)$ and $T \neq H_t$. If $T(-t-m, 1-N) < M+2$, then by Definition 2.2 (iii) and Equation (2.18),

$$m_T^* m_{H_t}^{*-1} \in \boxed{M+1}_{a\tau_{M+1}^{-1} q^{-2m}}^* \boxed{M+2}_{a\tau_{M+1}^{-1} q^{-2m}}^{*-1} \widehat{\mathcal{Q}}^- = A_{M+1, aq^{-2m-1}}^{-1} \widehat{\mathcal{Q}}^-.$$

If $T(-t-m, 1-N) = M+2$, then $T(-k, -l) = M+N+1-l$ for $1 \leq l < N$. Equation (2.18) implies that $m_T^* m_{H_t}^{*-1} A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \leq M$.

Lastly, viewing $D_{m,a}^{(M,s)}$ (after tensoring with a one-dimensional module) as a sub-quotient of $S_t \otimes W_{t+s, aq^{2t-1}}^{(M)} \otimes W_{m, aq^{2m}}^{(M-1)}$ and choosing $t \in \mathbb{Z}_{>0}$ so large that $\mathbf{n} \notin A_{M, aq^{2t}}^{-1} \widehat{\mathcal{Q}}^-$, we obtain $m_{H_t}^* \mathbf{n} \in \text{wt}_\ell(S_t)$, and so $\mathbf{n} A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \leq M$. From (ii)–(iii) it follows that $\mathbf{n} A_{M,a} = 1$.

It remains to show that $\mathbf{d}_{m,a}^{(M,s)} A_{M,a}^{-1} \in \text{wt}_\ell(D_{m,a}^{(M,s)})$. Indeed, as a $U_q(\mathfrak{g})$ -module, $D_{m,a}^{(M,s)}$ has a highest weight vector of highest weight $q^{m\varpi_{M-1} + s\varpi_M + m\varpi_{M+1}}$, and so $q^{m\varpi_{M-1} + s\varpi_M + m\varpi_{M+1} - \alpha_M} \in \text{wt}(D_{m,a}^{(M,s)})$. This means that there exists $\mathbf{n} \in (\mathbf{d}_{m,a}^{(M,s)})^{-1} \text{wt}_\ell(D_{m,a}^{(M,s)})$ with $\varpi(\mathbf{n}) = q^{-\alpha_M}$, which forces $\mathbf{n} = A_{M,a}^{-1}$. \square

As an illustration, for $\mathfrak{g} = \mathfrak{gl}(3|4)$ and $(m, s, t) = (2, 3, 1)$ we have

$$H = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & & \\ \hline \end{array} \in \mathcal{B}_+(3\varpi_3 + 2\varpi_2), \quad H_t = \begin{array}{|c|c|c|} \hline 5 & 6 & 7 \\ \hline 5 & 6 & 7 \\ \hline 4 & 5 & 6 & 7 \\ \hline \end{array} \in \mathcal{B}_-(3\varpi_3 + \varpi_1).$$

4. PROOF OF TQ RELATIONS: THEOREM 3.3

The crucial part in the proof is the irreducibility of arbitrary tensor products of positive pre-fundamental modules. In the case of quantum affine algebras this was proved in [23, Theorem 4.11] and [18, Lemma 5.1]. Our approach is similar to [18], based on the duality functor \mathbb{D} in Lemma 1.8.

Lemma 4.1. *Let $a \in \mathbb{C}^\times$ and $i \in I_0$. We have*

$$\chi_q(L_{i,a}^+) = \Psi_{i,a} \times \chi(L_{i,a}^+).$$

Proof. The idea is similar to [23, Theorem 4.1], using Corollary 2.10. The weights of $L_{i,a}^+$ are of the form $p = q^{-\sum_{i \in I_0} n_i \alpha_i} \in q^{\mathbf{Q}^-}$ with $n_i \in \mathbb{Z}_{\geq 0}$. We prove by induction on $\text{ht}(p) := \sum_{i \in I_0} n_i$ that $(L_{i,a}^+)_p$ is of ℓ -weight $\Psi_{i,a}p$. For $\text{ht}(p) = 0$ this is obvious as $(L_{i,a}^+)_1$ is spanned by a highest ℓ -weight vector.

Assume $\text{ht}(p) > 0$. Let $\mathbf{m}\Psi_{i,a}$ be an ℓ -weight of $(L_{i,a}^+)_p$ and write $\mathbf{m} = \mathbf{m}(z)$. We need to prove that $\mathbf{m}(z)$ is independent of z . Suppose not. The $\mathbf{m}(zq_i^{-2k}) \in \widehat{\mathfrak{P}}$ for $k \in \mathbb{Z}_{>0}$ are two-by-two distinct. Since $(L_{i,a}^+)_p$ is finite-dimensional, there exists $k_1 \in \mathbb{Z}_{>0}$ such that $\mathbf{m}(zq_i^{-2k})\Psi_{i,a} \notin \text{wt}_\ell(L_{i,a}^+)$ for all $k > k_1$. Equivalently, $\mathbf{m}\Psi_{i,aq_i^{2k}} \notin \text{wt}_\ell(L_{i,aq_i^{2k}}^+)$ for $k > k_1$ by using the pullback of $\Phi_{q_i^{2k}}$ in (1.1).

For $k > k_1$, let $\mathbf{n}_k := \Psi_{i,a}\Psi_{i,aq_i^{2k}}^{-1}(\varpi_{k,aq_i^{2k-1}}^{(i)})^{-1} \in \mathbf{R}$. Then $\mathbf{n}_k \equiv 1$. Viewing $L_{i,a}^+$ as a sub-quotient of $L_{i,aq_i^{2k}}^+ \otimes W_{k,aq_i^{2k-1}}^{(i)} \otimes L(\mathbf{n}_k)$, we obtain

$$\mathbf{m} = \mathbf{m}'_k \mathbf{m}''_k, \quad \mathbf{m}'_k \varpi_{i,aq_i^{2k-1}}^{(i)} \in \text{wt}_\ell(W_{k,aq_i^{2k-1}}^{(i)}), \quad \mathbf{m}''_k \Psi_{i,aq_i^{2k}} \in \text{wt}_\ell(L_{i,aq_i^{2k}}^+).$$

$\mathbf{m}'_k \neq \mathbf{m}$ since $\mathbf{m}\Psi_{i,aq_i^{2k}} \notin \text{wt}_\ell(L_{i,aq_i^{2k}}^+)$. This implies $\mathbf{m}'_k \neq 1$ and so $\mathbf{m}'_k \in A_{i,aq_i^{2k}}^{-1} \widehat{\mathcal{Q}}^-$ by Corollary 2.10. Let $p' := \varpi(\mathbf{m}''_k \Psi_{i,aq_i^{2k}}) \in \text{wt}(L_{i,aq_i^{2k}}^+)$. Then $\text{ht}(p') < \text{ht}(p)$. The induction hypothesis applied to p' shows that $\mathbf{m}''_k = \varpi(\mathbf{m}''_k) \in q^{\mathbf{Q}^-}$ and

$$\mathbf{m} \in A_{i,aq_i^{2k}}^{-1} \widehat{\mathcal{Q}}^- q^{\mathbf{Q}^-} \quad \text{for all } k \in \mathbb{Z}_{>k_1}.$$

This can not happen as the $A_{j,b}$ are free generators of the abelian group $\widehat{\mathcal{Q}}$. \square

Lemma 4.2. [51] *Let $a \in \mathbb{C}^\times$ and $i \in I_0$. When $m \rightarrow \infty$, $(\varpi_{m,a}^{(i)})^{-1} \chi_q(W_{m,a}^{(i)})$ converge, as formal power series in the $A_{j,b}^{-1}$ with $j \in I_0$ and $b \in aq^{\mathbb{Z}}$, to $\Psi_{i,aq_i} \chi_q(L_{i,aq_i}^-)$. In particular, $\Psi_{i,a} \text{wt}_\ell(L_{i,a}^-) \subseteq \{1\} \cup (A_{i,a}^{-1} \widehat{\mathcal{Q}}^-)$.*

Proof. This is [51, Lemma 6.6] on category $\mathcal{O}_{q^{-1}}$ of $Y_{q^{-1}}(\mathfrak{g})$ -modules. Up to tensor products by parity modules, the pullbacks by h of $L_{i,a\tau_i}^-$, $W_{m,a\tau_i q_i^{-1}}^{(i)}$ become $L_{i,a}^+$, $W_{m,a}^{(i)}$ in *loc. cit.* and respect the two q -character maps by Remark 1.2. \square

Corollary 4.3. *Any tensor product of positive (resp. negative) pre-fundamental modules in category \mathcal{O} is irreducible.*

Proof. Let $S = \otimes_{(i,a)} L_{i,a}^+$ be a tensor product of positive pre-fundamental modules and let $L(\widehat{p})$ be an arbitrary irreducible sub-quotient of S . We need to prove that $\widehat{p} = \prod_{(i,a)} \Psi_{i,a}$. By Lemma 4.1, $\widehat{p} \equiv \prod_{(i,a)} \Psi_{i,a}$.

By Lemma 1.8, the $Y_q(\mathfrak{g}')$ -module $\mathbb{D}(S)$ in category \mathcal{O}' contains an irreducible sub-quotient $\mathbb{D}(L(\widehat{p})) \cong L'(\widehat{p}')$ where $\widehat{p}'_i(z) = p_{\kappa+1-i}(z)$. Since $\widehat{p}' \equiv \prod_{(i,a)} \mathbf{m}_{i,a}$ with $\mathbf{m}_{i,a}$ being the highest ℓ -weight of $\mathbb{D}(L_{i,a}^+)$ and since $\mathbb{D}(L_{i,a}^+) \simeq L_{\kappa+1-i,aq^{N-M}}'^-$, we must have $\widehat{p}' = \prod_{(i,a)} \mathbf{m}_{i,a}$ and so $\widehat{p} = \prod_{(i,a)} \Psi_{i,a}$. \square

Proof of Theorem 3.3. Let $T := N_{i,a}^+ \otimes L_{i,a}^+$. We need to prove that T has exactly two irreducible sub-quotient $S' := L(\mathbf{n}_{i,a}^+ \Psi_{i,a})$ and $S'' := L(\mathbf{n}_{i,a}^+ \Psi_{i,a} A_{i,a}^{-1})$ of multiplicity one, from which follows Theorem 3.3 since S' and S'' are irreducible tensor products of positive pre-fundamental modules with D . The idea is similar to the proof of Corollary 3.6. For S' this is clear.

1. $W_{1,aq_i^{-1}}^{(i)}$ viewed as a sub-quotient of $N_{i,a}^+ \otimes (\otimes_{j \sim i} L_{j,aq_{ij}^{-1}}^-)$, we deduce from Corollary 2.10 that $A_{i,a}^{-1} = \mathbf{m}' \prod_{j \sim i} \mathbf{m}'_j$ where $\mathbf{m}' \mathbf{n}_{i,a}^+ \in \text{wt}_\ell(N_{i,a}^+)$ and $\mathbf{m}'_j \Psi_{j,aq_{ij}^{-1}}^{-1} \in \text{wt}_\ell(L_{j,aq_{ij}^{-1}}^-)$ for $j \sim i$. If $\mathbf{m}'_j \neq 1$, then $\varpi(\mathbf{m}'_j) \in q^{-\alpha_j + \mathbf{Q}^-}$ and so $\varpi(A_{i,a}^{-1}) = q^{-\alpha_i} \in q^{-\alpha_j + \mathbf{Q}^-}$ with $j \sim i$, contradiction. This implies $\mathbf{m}' = A_{i,a}^{-1}$ and $\mathbf{n}_{i,a}^+ \Psi_{i,a} A_{i,a}^{-1} \in \text{wt}_\ell(T) \setminus \text{wt}_\ell(S')$, so S'' must be an irreducible sub-quotient of T . By Corollary 4.3, $\chi_q(S') + \chi_q(S'') = \mathbf{n}_{i,a}^+ \Psi_{i,a} (1 + A_{i,a}^{-1}) \chi(L_{i,1}^+) \prod_{j \sim i} \chi(L_{j,1}^+)$.

2. Assume $\mathbf{n}_{i,a}^+ \mathbf{n} \in \text{wt}_\ell(N_{i,a}^+)$ and $\mathbf{n} \neq 1$. For $m \in \mathbb{Z}_{>0}$ let $S_m := L(\mathbf{n}_{i,a}^+ (\mathbf{d}_{m,a}^{(i,1)})^{-1})$ and view $N_{i,a}^+$ as a sub-quotient of $D_{m,a}^{(i,1)} \otimes S_m$. Write

$$\mathbf{n} = \mathbf{n}'_m \mathbf{n}''_m, \quad \mathbf{n}'_m \mathbf{d}_{m,a}^{(i,1)} \in \text{wt}_\ell(D_{m,a}^{(i,1)}), \quad \mathbf{n}''_m \mathbf{n}_{i,a}^+ (\mathbf{d}_{m,a}^{(i,1)})^{-1} \in \text{wt}_\ell(S_m).$$

By Remark 3.2, we have $\mathbf{n}_{i,a}^+ (\mathbf{d}_{m,a}^{(i,1)})^{-1} \equiv \prod_{j \sim i} \Psi_{j,aq_{ij}^{-2m-1}}$. It follows from Corollary 4.3 that $\mathbf{n}''_m \in q^{\mathbf{Q}^-}$, $\chi(S_m) = \prod_{j \sim i} \chi(L_{j,1}^+)$, and so $\mathbf{n} \in \widehat{\mathcal{Q}}^- q^{\mathbf{Q}^-}$.

Choose $t \in \mathbb{Z}_{>0}$ large enough so that $\mathbf{n} \in \widehat{\mathcal{Q}}_t^- q^{\mathbf{Q}^-}$ where $\widehat{\mathcal{Q}}_t^-$ is the submonoid of $\widehat{\mathcal{Q}}$ generated by the A_{j,aq_l}^{-1} with $-t < l < t$. Then for $m > t$, we must have $\mathbf{n}'_m \in \{1, A_{i,a}^{-1}\}$ by Corollaries 3.6–3.7. This implies that \mathbf{n}''_m is uniquely determined by \mathbf{n} and $\dim(N_{i,a}^+)_\mathbf{n} \leq \dim(S_m)_{\mathbf{n}''_m}$. As a consequence, the coefficient of any $\widehat{p} \in \widehat{\mathfrak{P}}$ in $\mathbf{n}_{i,a}^+ (1 + A_{i,a}^{-1}) \prod_{j \sim i} \chi(L_{j,1}^+) - \chi_q(N_{i,a}^+)$ is non-negative.

Combining 1 and 2, we obtain the q -character formulas

$$\chi_q(N_{i,a}^+) = \mathbf{n}_{i,a}^+ (1 + A_{i,a}^{-1}) \prod_{j \sim i} \chi(L_{j,1}^+)$$

and $\chi_q(S') + \chi_q(S'') = \chi_q(T)$, completing the proof of Theorem 3.3. \square

5. MAIN RESULT: ASYMPTOTIC TQ RELATIONS

We replace the L, N in Equation (3.27) by $U_q(\widehat{\mathfrak{g}})$ -modules using the functor \mathbb{D} .

Corollary 5.1. *Let $i \in I_0$ and $a \in \mathbb{C}^\times$. In the Grothendieck ring $K_0(\mathcal{O})$:*

$$(5.28) \quad [N_{i,a}^-][L_{i,a}^-] = [L_{i,aq_i^2}^-] \prod_{j \in I_0: j \sim i} [L_{j,aq_{ij}}^-] + [D][L_{i,aq_i^{-2}}^-] \prod_{j \in I_0: j \sim i} [L_{j,aq_{ij}^{-1}}^-]$$

where $D = L(\mathbf{n}_{i,a}^- \Psi_{i,a}^{-1} A_{i,a}^{-1} \Psi_{i,aq_i^{-2}} \prod_{j \sim i} \Psi_{j,aq_{ij}^{-1}})$ is one-dimensional.

Proof. Applying \mathbb{D}^{-1} to Equation (3.27) in $K_0(\mathcal{O}')$ gives Equation (5.28) by Lemma 1.8. Take q -characters in Equation (5.28). By Lemma 4.2, $\mathbf{n}_{i,a}^- \Psi_{i,a}^{-1} A_{i,a}^{-1}$ appears at the left-hand side, but in none of the $\chi_q(L_{j,b}^-)$ at the right-hand side. This forces $\chi_q(D) \Psi_{i,aq_i^{-2}}^{-1} \prod_{j \sim i} \Psi_{j,aq_{ij}^{-1}}^{-1} = \mathbf{n}_{i,a}^- \Psi_{i,a}^{-1} A_{i,a}^{-1}$ and proves the second statement. \square

Equation (5.28) becomes [33, Example 7.8] when $N = 0$.

Definition 5.2. Let $i \in I_0$ and $a, c \in \mathbb{C}^\times$. Set $\omega_{c,a}^{(i)} := [c]_i \frac{\Psi_{i,a,c-2}}{\Psi_{i,a}} \in \mathbf{R}_U$ and:

$$\mathbf{n}_{c,a}^{(i)} := \omega_{\hat{q}_i, a\hat{q}_i^2}^{(i)} \prod_{j \in I_0: j \sim i} \omega_{c_{ij}^{-1}, aq_{ij}}^{(j)}, \quad \mathbf{m}_{c,a}^{(i)} := \omega_{q_i, a}^{(i)} \prod_{j \in I_0: j \sim i} \omega_{c_{ij}^{-1}, aq_{ij}^{-1} c_{ij}^{-2}}^{(j)} \in \mathbf{R}_U.$$

Here $[c]_i := ((p_j)_{j \in I; \bar{0}}) \in \mathfrak{P}$ is: $\frac{p_j}{p_{j+1}} = c^{\delta_{ji}}$ for $j \in I_0$; if $i \leq M$ then $p_\kappa = 1$; if $i > M$ then $p_1 = 1$. For $i, j \in I_0$, set $c_{ij} := c^{(\alpha_i, \alpha_j)}$. Let $M_{c,a}^{(i)}$ be the irreducible $U_q(\widehat{\mathfrak{g}})$ -module in category \mathcal{O} of highest ℓ -weight $\mathbf{m}_{c,a}^{(i)}$.

Proposition 5.3. [51, §§6,8] For $i \in I_0$ and $a, c \in \mathbb{C}^\times$, there exists a $U_q(\widehat{\mathfrak{g}})$ -module $\mathcal{W}_{c,a}^{(i)}$ in category \mathcal{O} with a unique highest ℓ -weight vector (up to scalar) and

$$\chi_q(\mathcal{W}_{c,a}^{(i)}) = \omega_{c,a}^{(i)} \times \widetilde{\chi}_q(L_{i,a}^-).$$

If $c^2 \notin q^\mathbb{Z}$, then $\mathcal{W}_{c,a}^{(i)}$ is irreducible.

In [51, §4], $\mathcal{W}_{c,a}^{(i)}$, denoted by $W_{c,a}^{(i)}$, was an “asymptotic limit” of a special inductive system of vector superspaces $(W_{m,aq_i^{-1}}^{(i)})_{m \in \mathbb{Z}_{>0}}$.

Proposition 5.4. Let $i \in I_0$ and $a, c \in \mathbb{C}^\times$. There exists a $U_q(\widehat{\mathfrak{g}})$ -module $\mathcal{N}_{c,a}^{(i)}$ in category \mathcal{O} with a unique highest ℓ -weight vector (up to scalar) and

$$\chi_q(\mathcal{N}_{c,a}^{(i)}) = \mathbf{n}_{c,a}^{(i)} \times \widetilde{\chi}_q(N_{i,a}^-).$$

If $c^2 \notin q^\mathbb{Z}$, then $\mathcal{N}_{c,a}^{(i)}$ is irreducible.

The proof of this proposition will be given in Section 7 (it also works for Proposition 5.3). We are able to prove the main result of the paper.

Theorem 5.5. Let $i \in I_0$ and $a, c, d \in \mathbb{C}^\times$. In the Grothendieck ring $K_0(\mathcal{O})$:

$$(5.29) \quad [\mathcal{N}_{c,a}^{(i)}][\mathcal{W}_{d,a}^{(i)}] = [\mathcal{W}_{d\hat{q}_i, a\hat{q}_i^2}^{(i)}] \prod_{j \in I_0: j \sim i} [\mathcal{W}_{c_{ij}^{-1}, aq_{ij}}^{(j)}] \\ + [D_i^-][\mathcal{W}_{dq_i^{-1}, aq_i^{-2}}^{(i)}] \prod_{j \in I_0: j \sim i} [\mathcal{W}_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1}}^{(j)}]$$

where $D_i^- = L(\mathbf{n}_{c,a}^{(i)} \omega_{d,a}^{(i)} A_{i,a}^{-1} (\omega_{dq_i^{-1}, aq_i^{-2}}^{(i)} \prod_{j \sim i} \omega_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1}}^{(j)})^{-1})$ is a one-dimensional $U_q(\widehat{\mathfrak{g}})$ -module. If $c^2, d^2 \notin q^\mathbb{Z}$, then in $K_0(\mathcal{O})$

$$(5.30) \quad [M_{c,a}^{(i)}][\mathcal{W}_{d,ad^2}^{(i)}] = [\mathcal{W}_{dq_i, ad^2}^{(i)}] \prod_{j \in I_0: j \sim i} [\mathcal{W}_{c_{ij}^{-1}, aq_{ij}^{-1} c_{ij}^{-2}}^{(j)}] \\ + [D_i][\mathcal{W}_{d\hat{q}_i^{-1}, ad^2}^{(i)}] \prod_{j \in I_0: j \sim i} [\mathcal{W}_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1} c_{ij}^{-2}}^{(j)}]$$

with $D_i = L(\mathbf{m}_{c,a}^{(i)} \omega_{d,ad^2}^{(i)} A_{i,a}^{-1} (\omega_{d\hat{q}_i^{-1}, ad^2}^{(i)} \prod_{j \sim i} \omega_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1} c_{ij}^{-2}}^{(j)})^{-1})$ one-dimensional.

The advantage of Equation (5.30) over (5.29) is that for fixed $j \in I_0$ the spectral parameter a in $\mathcal{W}_{c,a}^{(j)}$ is also fixed. This is crucial in deriving BAE in Section 9.

Proof. D_i^- is one-dimensional by Definition 5.2 and Equation (2.26):

$$\mathbf{n}_{c,a}^{(i)} \omega_{d,a}^{(i)} A_{i,a}^{-1} \equiv \left(\frac{\Psi_{i,a}}{\Psi_{i,a\hat{q}_i^2}} \prod_{j \sim i} \frac{\Psi_{j,aq_{ij} c_{ij}^2}}{\Psi_{j,aq_{ij}}} \right) \times \frac{\Psi_{i,ad^{-2}}}{\Psi_{i,a}} \times \left(\frac{\Psi_{i,aq_i^{-2}}}{\Psi_{i,a\hat{q}_i^2}} \prod_{j \sim i} \frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}}} \right)^{-1} \\ \equiv \frac{\Psi_{i,ad^{-2}}}{\Psi_{i,aq_i^{-2}}} \prod_{j \sim i} \frac{\Psi_{j,aq_{ij} c_{ij}^2}}{\Psi_{j,aq_{ij}^{-1}}} \equiv \omega_{dq_i^{-1}, aq_i^{-2}}^{(i)} \prod_{j \in I_0: j \sim i} \omega_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1}}^{(j)}.$$

Dividing the q -characters of both sides of (5.29) by $\mathbf{n}_{c,a}^{(i)} \omega_{d,a}^{(i)}$, we obtain the normalized q -characters of (5.28) by Propositions 5.3–5.4. This proves (5.29).

As in Table (1.14), let $\mathcal{N}_{c,a}^{(i)}, \mathcal{W}_{c,a}^{(i)}$ be the corresponding $U_q(\widehat{\mathfrak{g}}')$ -modules in category \mathcal{O}' . Since $c^2, d^2 \notin q^\mathbb{Z}$, by Propositions 5.3–5.4 and Lemma 1.8, $\mathbb{D}(M_{c,a}^{(i)}) \simeq \mathcal{N}_{c,aq^{N-M}}^{(M+N-i)}$ and $\mathbb{D}(\mathcal{W}_{c,a}^{(i)}) \simeq \mathcal{W}_{c^{-1}, ac^{-2} q^{N-M}}^{(M+N-i)}$ as irreducible $U_q(\widehat{\mathfrak{g}}')$ -modules in category \mathcal{O}' . Applying \mathbb{D}^{-1} to (5.29) in $K_0(\mathcal{O}')$ gives (5.30). The ℓ -weight of D_i is fixed similarly as in the proof of Corollary 5.1. \square

6. CYCLICITY OF TENSOR PRODUCTS

We provide a criteria for a tensor product of Kirillov–Reshetikhin modules to be of highest ℓ -weight, which is needed to prove Theorem 3.4 and Proposition 5.4.

For $i, j \in \mathbb{Z}_{>0}$ let us define the q -segment

$$\mathcal{S}(i, j) := \{q^{-i-j+2r} \mid 0 \leq r < \min(i, j)\} \subset \mathbb{C}^\times.$$

It is $q^{j-i}\Sigma(i, j)^{-1}$ in [50, §5] and is symmetric in i, j .

Theorem 6.1. *Let $s \in \mathbb{Z}_{>0}$. For $1 \leq l \leq s$ let $1 \leq i_l \leq M$ and $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$. The $U_q(\widehat{\mathfrak{g}})$ -module $W_{m_1, a_1}^{(i_1)} \otimes W_{m_2, a_2}^{(i_2)} \otimes \cdots \otimes W_{m_s, a_s}^{(i_s)}$ is of highest ℓ -weight if*

$$(6.31) \quad \frac{a_j}{a_k} \notin \bigcup_{p=1}^{m_j} q^{2p-2m_k} \mathcal{S}(i_j, i_k) \quad \text{for } 1 \leq j < k \leq s.$$

The idea is similar to [49, 50], which in turn was inspired by [11], by restricting to diagram subalgebras. Let A, B be Hopf superalgebras and let $\iota : A \rightarrow B$ be a morphism of superalgebras. If W is a B -module and W' is a sub- A -module of the A -module $\iota^*(W)$, then let $\iota^\bullet(W')$ denote the A -module structure on W' .

For $1 \leq p \leq 3$, define the quantum affine superalgebra U_p with RTT generators $s_{ij;p}^{(n)}, t_{ij;p}^{(n)}$ and the superalgebra morphism $\iota_p : U_p \rightarrow U_q(\widehat{\mathfrak{g}})$ as follows: $U_1 := U_q(\widehat{\mathfrak{gl}(1|1)})$, $U_2 := U_{q^{-1}}(\widehat{\mathfrak{gl}(1|1)})$ and $U_3 := U_q(\widehat{\mathfrak{gl}(M-1|N)})$, so that in $s_{ij;p}^{(n)}, t_{ij;p}^{(n)}$ we understand either $(1 \leq i, j, p \leq 2)$ or $(1 \leq i, j < M+N, p=3)$;

$$\begin{aligned} \iota_1 : U_1 &\rightarrow U_q(\widehat{\mathfrak{g}}), & s_{ij;1}^{(n)} &\mapsto s_{i'j'}^{(n)}, & t_{ij;1}^{(n)} &\mapsto t_{i'j'}^{(n)}; \\ \iota_2 : U_2 &\rightarrow U_q(\widehat{\mathfrak{g}}), & s_{ij;2}^{(n)} &\mapsto h(\bar{s}_{i'j'}^{(n)}), & t_{ij;2}^{(n)} &\mapsto h(\bar{t}_{i'j'}^{(n)}); \\ \iota_3 : U_3 &\rightarrow U_q(\widehat{\mathfrak{g}}), & s_{ij;3}^{(n)} &\mapsto s_{i+1, j+1}^{(n)}, & t_{ij;3}^{(n)} &\mapsto t_{i+1, j+1}^{(n)}. \end{aligned}$$

Here h is the involution in Equation (1.3) and $1' = 1, 2' = M+N$.

Lemma 6.2. [49, Lemma 3.7] *The tensor product of a lowest ℓ -weight $U_q(\widehat{\mathfrak{g}})$ -module with a highest ℓ -weight module is generated, as a $U_q(\widehat{\mathfrak{g}})$ -module, by a tensor product of a lowest ℓ -weight vector with a highest ℓ -weight vector.*

Let $1 \leq p \leq 2$. We recall the notion of *Weyl module* over U_p from [50]. Let $f(z) \in \mathbb{C}(z)$ be a product of the $c \frac{1-za}{1-zac^2}$ with $a, c \in \mathbb{C}^\times$ and let $P(z) \in 1 + z\mathbb{C}[z]$ be such that $\frac{P(z)}{f(z)} \in \mathbb{C}[z]$. The Weyl module $\mathcal{W}_p(f; P)$ is the U_p -module generated by a highest ℓ -weight vector w of even parity such that

$$s_{11;p}(z)w = f(z)w = t_{11;p}(z)w, \quad s_{22;p}(z)w = w = t_{22;p}(z)w,$$

and $\frac{P(z)}{f(z)} s_{21;p}(z)w$, as a formal power series in z with coefficients in $\mathcal{W}_p(f; P)$, is a polynomial in z of degree $\leq \deg P$. Given another pair (g, Q) , if the polynomials $\frac{P(z)}{f(z)}$ and $Q(z)$ are co-prime, then $\mathcal{W}_p(f; P) \otimes \mathcal{W}_p(g; Q)$ is a quotient of $\mathcal{W}_p(fg; PQ)$ and is of highest ℓ -weight; see [50, Proposition 14].

Example 6.3. In the situation of Theorem 6.1, fix $v_l \in W_{m_l, a_l}^{(i_l)}$ a highest ℓ -weight vector. Let W_p be the sub- U_p -module of $\iota_p^*(\otimes_{l=1}^s W_{m_l, a_l}^{(i_l)})$ generated by $\otimes_{l=1}^s v_l$. Then $\iota_p^\bullet(W_p)$ is a quotient of the Weyl module \mathcal{W}_p for $1 \leq p \leq 2$ where

$$\begin{aligned} \mathcal{W}_1 &:= \mathcal{W}_1 \left(\prod_{l=1}^s \frac{q^{m_l} - za_l q^{M-N-i_l-m_l}}{1 - za_l q^{M-N-i_l}}; \prod_{l=1}^s (1 - za_l q^{M-N-i_l-2m_l}) \right), \\ \mathcal{W}_2 &:= \mathcal{W}_2 \left(\prod_{l=1}^s \frac{q^{-m_l} - za_l q^{N-M+i_l-m_l}}{1 - za_l q^{N-M+i_l-2m_l}}; \prod_{l=1}^s (1 - za_l q^{N-M+i_l}) \right). \end{aligned}$$

$\iota_3^\bullet(W_3)$ is the tensor product $\otimes_{l=1}^s W_{m_l, a_l}^{3(i_l-1)}$ of KR modules over U_3 . The proof is the same as [50, Lemmas 18 & 20], based on Corollary 2.10.

For $p \in \mathbb{Z}_{>0}$, let $\mathfrak{g}_p := \mathfrak{gl}(1|p)$ and let $U_q(\widehat{\mathfrak{g}}_p)$ be the quantum affine superalgebra with RTT generators $s_{ij|p}^{(n)}, t_{ij|p}^{(n)}$ for $1 \leq i, j \leq p+1$. Similarly $U_{q^{-1}}(\widehat{\mathfrak{g}}_p)$ with RTT generators $\bar{s}_{ij|p}^{(n)}, \bar{t}_{ij|p}^{(n)}$ and the involution $h_p : U_{q^{-1}}(\widehat{\mathfrak{g}}_p) \longrightarrow U_q(\widehat{\mathfrak{g}}_p)$ are defined. For $1 \leq p \leq N$, the following extends uniquely to a superalgebra morphism

$$(6.32) \quad \vartheta_p : U_q(\widehat{\mathfrak{g}}_p) \longrightarrow U_q(\widehat{\mathfrak{g}}) : \quad s_{ij|p}^{(n)} \mapsto s_{i'j'}^{(n)}, \quad t_{ij|p}^{(n)} \mapsto t_{i'j'}^{(n)}$$

where $i' = 1$ and $j' = M + N - p - 1 + i$ for $2 \leq i \leq p+1$.

Definition 6.4. Let $s \in \mathbb{Z}_{>0}$ and $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$ for $1 \leq l \leq s$. The Weyl module $\mathcal{W}^p(\prod_{l=1}^s \varpi_{m_l, a_l})$ is the $U_q(\widehat{\mathfrak{g}}_p)$ -module generated by a highest ℓ -weight vector w of even parity such that for $2 \leq j \leq p+1$,

$$\begin{aligned} s_{11|p}(z)w &= w \prod_{l=1}^s \frac{q^{m_l} - za_l q^{-p-m_l}}{1 - za_l q^{-p}} = t_{11|p}(z)w, \\ h_p(\bar{s}_{11|p}(z))w &= w \prod_{l=1}^s \frac{q^{-m_l} - za_l q^{p-m_l}}{1 - za_l q^{p-2m_l}} = h_p(\bar{t}_{11|p}(z))w, \\ s_{jj|p}(z)w &= t_{jj|p}(z)w = h_p(\bar{s}_{jj|p}(z))w = h_p(\bar{t}_{jj|p}(z))w = w, \end{aligned}$$

and the following vector-valued polynomials in z are of degree $\leq s$:

$$\prod_{l=1}^s (1 - za_l q^{-p}) \times s_{j1|p}(z)w, \quad \prod_{l=1}^s (1 - za_l q^{p-2m_l}) \times h_p(\bar{s}_{j1|p}(z))w.$$

Let $L^p(\prod_{l=1}^s \varpi_{m_l, a_l})$ denote its irreducible quotient of $\mathcal{W}^p(\prod_{l=1}^s \varpi_{m_l, a_l})$.

Example 6.5. Let $1 \leq p \leq N$. In Example 6.3, let W^p be the sub- $U_q(\widehat{\mathfrak{g}}_p)$ -module of $\vartheta_p^*(\otimes_{l=2}^s W_{m_l, a_l}^{(i_l)})$ generated by $\otimes_{l=2}^s v_l$. Then $\vartheta_p^\bullet(W^p)$ is a quotient of the Weyl module $\mathcal{W}^p(\prod_{l=2}^s \varpi_{m_l, a_l q^{M-N-i_l+p}})$ over $U_q(\widehat{\mathfrak{g}}_p)$.

Example 6.6. Suppose $m_1 \leq N$ and take $p = m_1$. In $W_{m_1, a_1}^{(i_1)}$ there is a non-zero vector v_1^1 whose ℓ -weight corresponds to the tableau $T_1^1 \in \mathcal{B}_-(m_1 \varpi_{i_1})$ such that: $T_1^1(-i_1, -j) = 1$ for $1 \leq j \leq m_1$ and $T_1^1(-i, -j) = N + M - j + 1$ for $1 \leq i < i_1$ and $1 \leq j \leq m_1$. Let X be the sub- $U_q(\widehat{\mathfrak{g}}_{m_1})$ -module of $\vartheta_{m_1}^*(W_{m_1, a_1}^{(i_1)})$ generated by v_1^1 . By comparing the character formulas in Remark 2.5 we see that the $U_q(\widehat{\mathfrak{g}}_{m_1})$ -module $\vartheta_{m_1}^\bullet(X)$ is irreducible and in terms of evaluation modules:

$$\begin{aligned} \vartheta_{m_1}^\bullet(X) &\cong V_q^+(m_1 \epsilon_1 + \sum_{j=1}^{m_1} (i_1 - 1) \epsilon_{j+1}; a_1 q^{M-N-i_1}) \\ &\simeq V_q^+((m_1 + i_1 - 1) \epsilon_1; a_1 q^{M-N+i_1-2}) \cong V_q^-((m_1 + i_1 - 1) \epsilon_1; a_1 q^{M-N-i_1}) \\ &\cong L^{m_1}(\varpi_{m_1+i_1-1, a_1 q^{M-N+i_1+m_1-2}}). \end{aligned}$$

Let v_1^2 be a lowest ℓ -weight vector of the $U_q(\widehat{\mathfrak{g}}_{m_1})$ -module $\vartheta_{m_1}^\bullet(X)$. Then v_1^2 corresponds to the tableau $T_1^2 \in \mathcal{B}_-(m_1 \varpi_{i_1})$ such that $T_1^2(-i, -j) = N + M - j + 1$ for $1 \leq i \leq i_1$ and $1 \leq j \leq m_1$; it is a lowest ℓ -weight vector of the $U_q(\widehat{\mathfrak{g}})$ -module $W_{m_1, a_1}^{(i_1)}$. Notice that $s_{ij}^{(n)} X = 0$ if $2 \leq j \leq M + N - m_1$. Combining with Example 6.5, we observe that $X \otimes W^{m_1}$ is stable by $\vartheta_{m_1}(U_q(\widehat{\mathfrak{g}}_{m_1}))$ and the identity map is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{m_1})$ -modules $\vartheta_{m_1}^\bullet(X \otimes W^{m_1}) \cong \vartheta_{m_1}^\bullet(X) \otimes \vartheta_{m_1}^\bullet(W^{m_1})$.

Lemma 6.7. Let $p, s \in \mathbb{Z}_{>0}$ and let $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$ for $1 \leq l \leq s$. Assume $m_1 \geq p$. The $U_q(\widehat{\mathfrak{g}}_p)$ -module $L^p(\varpi_{m_1, a_1}) \otimes \mathcal{W}^p(\prod_{l=2}^s \varpi_{m_l, a_l})$ is of highest ℓ -weight if $a_1 \neq a_l q^{2t-2m_1-2}$ for $2 \leq l \leq s$ and $1 \leq t \leq p$.

Proof. By induction on p : for $p = 1$ we are led to consider the tensor product

$$\mathcal{W}_1 \left(\frac{q^{m_1} - za_1q^{-1-m_1}}{1 - za_1q^{-1}}; 1 - za_1q^{-1-2m_1} \right) \otimes \mathcal{W}_1 \left(\prod_{l=2}^s \frac{q^{m_l} - za_lq^{-1-m_l}}{1 - za_lq^{-p}}; \prod_{l=2}^s (1 - za_lq^{-1-2m_l}) \right)$$

of Weyl modules over $U_1 = U_q(\widehat{\mathfrak{g}}_1)$, which is of highest ℓ -weight if $a_1 \neq a_lq^{-2m_l}$ for $2 \leq l \leq s$. Assume $p > 1$. Consider $\vartheta_{p-1} : U_q(\widehat{\mathfrak{g}}_{p-1}) \rightarrow U_q(\widehat{\mathfrak{g}}_p)$ in Equation (6.32) with $\mathfrak{g} = \mathfrak{gl}(1|p)$. Let $v_1 \in L^p(\varpi_{m_1, a_1})$ and $w \in \mathcal{W}^p(\prod_{l=2}^s \varpi_{m_l, a_l})$ be highest ℓ -weight vectors of the corresponding $U_q(\widehat{\mathfrak{g}}_p)$ -modules and let

$$X_1 := \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))v_1, \quad Y_1 := \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))w.$$

Using evaluation modules over $U_q(\widehat{\mathfrak{g}}_p)$ we have by Corollary 2.10 and Definition 6.4:

$$L^p(\varpi_{m_1, a_1}) \cong V_q^+(m_1\epsilon_1; a_1q^{-p}) \cong V_q^-(m_1\epsilon_1; a_1q^{p-2m_1}).$$

It follows that $s_{i2|p}^{(n)}X_1 = 0$ if $i \neq 2$. This implies that $X_1 \otimes Y_1$ is stable by $\vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))$ and the identity map is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{p-1})$ -modules:

$$\vartheta_{p-1}^\bullet(X_1 \otimes Y_1) \cong \vartheta_{p-1}^\bullet(X_1) \otimes \vartheta_{p-1}^\bullet(Y_1).$$

As in Example 6.6, the $U_q(\widehat{\mathfrak{g}}_{p-1})$ -module $\vartheta_{p-1}^\bullet(X_1)$ is irreducible and isomorphic to $L^{p-1}(\varpi_{m_1, a_1q^{-1}})$. By Definition 6.4, $\vartheta_{p-1}^\bullet(Y_1)$ is a quotient of the Weyl module $\mathcal{W}^{p-1}(\prod_{l=2}^s \varpi_{m_l, a_lq^{-1}})$. The induction hypothesis applied to $p-1$ shows that $L^{p-1}(\varpi_{m_1, a_1q^{-1}}) \otimes \mathcal{W}^{p-1}(\prod_{l=2}^s \varpi_{m_l, a_lq^{-1}})$ and so $\vartheta_{p-1}^\bullet(X_1) \otimes \vartheta_{p-1}^\bullet(Y_1)$ are of highest ℓ -weight. Let v'_1 be the lowest ℓ -weight vector of the $U_q(\widehat{\mathfrak{g}}_{p-1})$ -module $\vartheta_{p-1}^\bullet(X_1)$; it corresponds to the tableau $T \in \mathcal{B}_-(m_1\epsilon_1)$ such that $T(-1, -j) = p+2-j$ for $1 \leq j \leq p-1$ and $T(-1, -j) = 1$ for $p \leq j \leq m_1$. We have

$$(*) \quad v'_1 \otimes w \in \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))(v_1 \otimes w) = X_1 \otimes Y_1.$$

Notice that $s_{ij;2}^{(n)} \mapsto h_p(\overline{s}_{ij|p}^{(n)})$ and $t_{ij;2}^{(n)} \mapsto h_p(\overline{t}_{ij|p}^{(n)})$ extend uniquely to a superalgebra morphism $\iota : U_2 \rightarrow U_q(\widehat{\mathfrak{g}}_p)$. Let $X_2 := \iota(U_2)v'_1$ and $Y_2 := \iota(U_2)w$. The identification $L^p(\varpi_{m_1, a_1}) \cong V_q^-(m_1\epsilon_1; a_1q^{p-2m_1})$ gives $X_2 := \mathbb{C}v'_1 + \mathbb{C}v''_1$ where v''_1 is a lowest ℓ -weight vector of $L^p(\varpi_{m_1, a_1})$. This implies $h_p(\overline{s}_{ij|p}^{(n)})X_2 = 0$ if $i \notin \{1, 2\}$, meaning that $X_2 \otimes Y_2$ is stable by $\iota(U_2)$ and the graded permutation map is an isomorphism of U_2 -modules $\iota^\bullet(X_2 \otimes Y_2) \cong \iota^\bullet(Y_2) \otimes \iota^\bullet(X_2)$. By Definition 6.4 the tensor product $\iota^\bullet(Y_2) \otimes \iota^\bullet(X_2)$ of U_2 -modules is a quotient of

$$\mathcal{W}_2 \left(\prod_{l=2}^s \frac{q^{-m_l} - za_lq^{p-m_l}}{1 - za_lq^{p-2m_l}}; \prod_{l=2}^s (1 - za_lq^p) \right) \otimes \mathcal{W}_2 \left(\frac{q^{-m_1+p-1} - za_1q^{-m_1+1}}{1 - za_1q^{p-2m_1}}; 1 - za_1q^{2-p} \right),$$

which is of highest ℓ -weight since $a_lq^{p-2m_l} \neq a_1q^{2-p}$ for $2 \leq l \leq s$. The U_2 -module $\iota^\bullet(X_2 \otimes Y_2)$ is of highest ℓ -weight and $v''_1 \otimes w \in \iota(U_2)(v'_1 \otimes w)$, which together with $(*)$ implies $v''_1 \otimes w \in U_q(\widehat{\mathfrak{g}}_p)(v_1 \otimes w)$. The $U_q(\widehat{\mathfrak{g}}_p)$ -module $L^p(\varpi_{m_1, a_1})$ being generated by the lowest ℓ -weight vector v''_1 , we conclude by Lemma 6.2. \square

For $\mathfrak{gl}(1|3)$ we related the highest/lowest ℓ -weight vectors of $L^3(\varpi_{5,a})$ by:

$$v_1 = \boxed{1 \mid 1 \mid 1 \mid 1 \mid 1} \xrightarrow{\vartheta_2: (134)q} v'_1 = \boxed{1 \mid 1 \mid 1 \mid 3 \mid 4} \xrightarrow{\iota: (12)q^{-1}} v''_1 = \boxed{1 \mid 1 \mid 2 \mid 3 \mid 4}.$$

Proof of Theorem 6.1. Let us assume first that $m_l \leq N$ for all $1 \leq l \leq s$. We use a double induction on (M, s) with Lemma 6.7 being the initial case $M = 1$. Under Condition (6.31), the induction hypothesis on M applied to the tensor product

of KR modules over U_3 in Example 6.3 shows that $\iota_3^\bullet(W_3)$ is of highest ℓ -weight and $v_1^1 \otimes (\otimes_{l=2}^s v_l) \in \iota_3(U_3)(\otimes_{l=1}^s v_l)$. It suffices to prove that the $U_q(\widehat{\mathfrak{g}}_{m_1})$ -module $\vartheta_{m_1}^\bullet(X) \otimes \vartheta_{m_1}^\bullet(W^{m_1})$ in Example 6.6 is of highest ℓ -weight, from which follows $v_1^2 \otimes (\otimes_{l=2}^s v_l) \in \vartheta_{m_1}(U_q(\widehat{\mathfrak{g}}_{m_1}))\iota_3(U_3)(\otimes_{l=1}^s v_l)$. The $U_q(\widehat{\mathfrak{g}})$ -module $W_{m_1, a_1}^{(i_1)}$ being generated by the lowest ℓ -weight vector v_1^2 , we can use the second induction on s and Lemma 6.2 to conclude.

By Examples 6.5 and 6.6, $\vartheta_{m_1}^\bullet(X) \otimes \vartheta_{m_1}^\bullet(W^{m_1})$ is, up to tensor product by one-dimensional modules, a quotient of the $U_q(\widehat{\mathfrak{g}}_{m_1})$ -module

$$L^{m_1}(\varpi_{m_1+i_1-1, a_1 q^{M-N+i_1+m_1-2}}) \otimes W^{m_1} \left(\prod_{l=1}^s \varpi_{m_l, a_l q^{M-N-i_l+m_1}} \right),$$

which by Lemma 6.7 is of highest ℓ -weight if for $2 \leq l \leq s$ and $1 \leq t \leq m_1$:

$$a_1 q^{M-N+i_1+m_1-2} \neq a_l q^{M-N-i_l+m_1} \times q^{2t-2-2m_l},$$

namely, $a_1 \neq a_l q^{2t-2m_l-i_1-i_l}$. This is included in Condition (6.31).

Suppose $m_l > N$ for some $1 \leq l \leq s$. Let $m := \max(m_l : 1 \leq l \leq s)$ and let $U_4 := U_q(\widehat{\mathfrak{gl}}(M|N+m))$ be the quantum affine superalgebra with RTT generators $s_{ij}^{(n)}, t_{ij;4}^{(n)}$ for $1 \leq i, j \leq M+N+m$. There is a unique superalgebra morphism

$$\iota_4 : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_4, \quad s_{ij}^{(n)} \mapsto s_{ij;4}^{(n)}, \quad t_{ij}^{(n)} \mapsto t_{ij;4}^{(n)}.$$

Under Condition (6.31), the tensor product $\otimes_{l=1}^s W_{m_l, a_l}^{4(i_l)}$ of KR modules over U_4 is of highest ℓ -weight. For $1 \leq l \leq s$, let $X_l := \iota_4(U_q(\widehat{\mathfrak{g}}))v_l$ where $v_l \in W_{m_l, a_l}^{4(i_l)}$ is a highest ℓ -weight vector. Then a weight argument and Corollary 2.10 show that

$$\iota_4(U_q(\widehat{\mathfrak{g}}))(\otimes_{l=1}^s v_l) = \otimes_{l=1}^s X_l,$$

and as $U_q(\widehat{\mathfrak{g}})$ -modules $\iota_4^\bullet(\otimes_{l=1}^s X_l) \cong \otimes_{l=1}^s W_{m_l, a_l q^{-m}}^{(i_l)}$. This implies that the $U_q(\widehat{\mathfrak{g}})$ -module $\otimes_{l=1}^s W_{m_l, a_l q^{-m}}^{(i_l)}$ is of highest ℓ -weight, proving the theorem. \square

For $\mathfrak{gl}(3|6)$ we related the highest/lowest ℓ -weight vectors of $W_{4,a}^{(3)}$ by:

$$v_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array} \xrightarrow{\iota_3:(23456789)q} v_1^1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 6 & 7 & 8 & 9 \\ \hline 6 & 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\vartheta_4:(16789)q} v_1^2 = \begin{array}{|c|c|c|c|} \hline 6 & 7 & 8 & 9 \\ \hline 6 & 7 & 8 & 9 \\ \hline 6 & 7 & 8 & 9 \\ \hline \end{array}.$$

For $\lambda \in \mathcal{P}$ and $a \in \mathbb{C}^\times$ define the $U_{q^{-1}}(\widehat{\mathfrak{g}})$ -module $V_{q^{-1}}^+(\lambda; a)$ to be the pullback of the $U_{q^{-1}}(\mathfrak{g})$ -module $V_{q^{-1}}(\lambda)$ by $\overline{\text{ev}}_a^+$, as in Theorem 2.4. By Equation (1.6),

$$h^*(V_{q^{-1}}^-(\lambda; a)) \cong V_{q^{-1}}^+(\lambda; a).$$

Corollary 6.8. *The tensor product in Theorem 6.1 is of highest ℓ -weight if*

$$(6.33) \quad \frac{a_j}{a_k} \notin \bigcup_{p=1}^{m_k} q^{2p-2m_k} \mathcal{S}(i_j, i_k) \quad \text{for } 1 \leq j < k \leq s.$$

Proof. The tensor product T in Theorem 6.1 is of highest ℓ -weight if and only if so is the $U_{q^{-1}}(\widehat{\mathfrak{g}})$ -module $h^*(T)$. By Corollary 2.10 we have

$$h^*(T) \cong \otimes_{l=s}^1 V_{q^{-1}}^+(m_l \varpi_{i_l}; a_l q^{N-M-2m_l+i_l}).$$

Applying Theorem 6.1 to $U_{q^{-1}}(\widehat{\mathfrak{g}})$, by viewing $W_{m,a}^{(i)}$ first as $V_q^+(m\varpi_i; aq^{M-N-i})$ and then as $V_{q^{-1}}^+(m\varpi_i; aq^{N-M+i})$, we have that $h^*(T)$ is of highest ℓ -weight if

$$\frac{a_k q^{-2m_k}}{a_j q^{-2m_j}} \notin \bigcup_{p=1}^{m_k} q^{-2p+2m_j} \mathcal{S}(i_k, i_j)^{-1} \quad \text{for } 1 \leq j < k \leq s.$$

This is Condition (6.33) since $\mathcal{S}(i_k, i_j) = \mathcal{S}(i_j, i_k)$. \square

Let V be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module. Its *twisted dual* is the dual space $\text{Hom}_{\mathbb{C}}(V, \mathbb{C}) =: V^\vee$ endowed with the $U_q(\widehat{\mathfrak{g}})$ -module structure [50, §6]:

$$\langle x\varphi, v \rangle := (-1)^{|\varphi||x|} \langle \varphi, \mathbb{S}\Psi(x) \rangle \quad \text{for } x \in U_q(\widehat{\mathfrak{g}}), \varphi \in V^\vee, v \in V.$$

By Equation (1.2), $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$ if W is another finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module. V is irreducible if and only if both V and V^\vee are of highest ℓ -weight.

We recall the notion of *fundamental representations* from [50]. Let $1 \leq r \leq M$ and $1 \leq s < N$. Define (compare [50, Lemmas 5 & 6] with Corollary 2.10)

$$(6.34) \quad V_{r,a}^+ := W_{1,aq^{N-M-r}}^{(r)}, \quad V_{s,a}^- := W_{1,aq^{s+2}}^{(M+N-s)}, \quad V_{N,a}^- := W_{1,aq^{N+2}}^{(M-)}.$$

Lemma 6.9. *Let $1 \leq i \leq M < j < M + N$ and $(m, a) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$. We have:*

$$(W_{m,a}^{(i)})^\vee \simeq W_{m,a^{-1}q^{2m}}^{(i)}, \quad (W_{m,a}^{(j)})^\vee \simeq W_{m,a^{-1}q^{4-2m}}^{(j)}, \quad (W_{m,a}^{(M-)})^\vee \simeq W_{m,a^{-1}q^{4-2m}}^{(M-)}.$$

Proof. The twisted dual of a fundamental module is known [50, Lemma 27]:

$$(V_{i,a}^+)^\vee \simeq V_{i,a^{-1}q^{2(M-N+i+1)}}^+, \quad (V_{M+N-j,a}^-)^\vee \simeq V_{M+N-j;a^{-1}q^{-2(M+N+1-j)}}^-.$$

By Equation (6.34), $(W_{1,a}^{(i)})^\vee \cong W_{1,a^{-1}q^2}^{(i)}$ and $(W_{1,a}^{(j)})^\vee \cong W_{m,a^{-1}q^2}^{(j)}$. Viewing $W_{m,a}^{(i)}$ as the unique irreducible sub-quotient of $\otimes_{l=1}^m W_{1,aq_i^{2-2l}}^{(i)}$ of highest ℓ -weight $\varpi_{m,a}^{(i)}$, and taking twisted duals, we obtain the desired formulas. \square

Corollary 6.10. *Let $1 < i < M$, $a \in \mathbb{C}^\times$ and $m \in \mathbb{Z}_{>0}$. The $U_q(\widehat{\mathfrak{g}})$ -module $W_{m,a}^{(i-1)} \otimes W_{m,a}^{(i+1)}$ is irreducible.*

Proof. The tensor product and its twisted dual, which is $\simeq W_{m,a^{-1}q^{2m}}^{(i-1)} \otimes W_{m,a^{-1}q^{2m}}^{(i+1)}$ by Lemma 6.9, satisfy Condition (6.33) and are of highest ℓ -weight. \square

The following special result on Dynkin node M is needed in Section 7.

Lemma 6.11. [50] *For $m \in \mathbb{Z}_{>0}$, the $U_q(\widehat{\mathfrak{g}})$ -module $V_{N,aq^{-3}}^- \otimes (\otimes_{l=1}^m V_{N-1,aq^{2l-1}}^-) \otimes (\otimes_{k=1}^m V_{M-1,aq^{2M-2k-1}}^+)$ is of highest ℓ -weight. Moreover for $1 \leq k, l \leq m$ we have $V_{N-1,aq^{2l-1}}^- \otimes V_{M-1,aq^{2M-2k-1}}^+ \cong V_{M-1,aq^{2M-2k-1}}^+ \otimes V_{N-1,aq^{2l-1}}^-$.*

Proof. The first statement is induced from [50, Theorem 15] by the involution h as in [50, Remarks 3 & 4], and the second is a particular case of [50, Example 5]. \square

7. ASYMPTOTIC REPRESENTATIONS

In this section we construct the $U_q(\widehat{\mathfrak{g}})$ -module $\mathcal{N}_{c,a}^{(i)}$ of Proposition 5.4 for $i \in I_0$ and $a, c \in \mathbb{C}^\times$ from finite-dimensional representations.

For $m \in \mathbb{Z}_{>0}$, let $N_{m,a}^{(i)} := L(\mathbf{n}_{q^m,a}^{(i)})$ be the irreducible $U_q(\widehat{\mathfrak{g}})$ -module; it is finite-dimensional by Lemma 1.5 (4). Fix $v^m \in N_{m,a}^{(i)}$ to be a highest ℓ -weight vector.

The main step is to construct an inductive system $(N_{m,a}^{(i)})_{m \in \mathbb{Z}_{>0}}$ compatible with (normalized) q -characters, as in [32, §4.2] and [33, Theorem 7.6]. We shall need the cyclicity results in Section 6 to adapt the arguments of [32, 33].

Lemma 7.1. *If $\mathbf{n}_{q^m,a}^{(i)} \mathbf{m} \in \text{wt}_\ell(N_{m,a}^{(i)})$, then $\mathbf{n}_{i,a}^- \mathbf{m} \in \text{wt}_\ell(N_{i,a}^-)$ and*

$$\dim(N_{m,a}^{(i)})_{\mathbf{n}_{q^m,a}^{(i)} \mathbf{m}} \leq \dim(N_{i,a}^-)_{\mathbf{n}_{i,a}^- \mathbf{m}}.$$

Proof. The first paragraph of the proof of [33, Theorem 7.6] can be copied here, based on Lemma 4.1 and the fact that \mathbf{m} is a product of the $A_{j,b}^{-1}$. To observe the latter fact, we can realize $N_{m,a}^{(i)}$ as a tensor product of KR modules with one-dimensional modules and apply Theorem 2.4. \square

Lemma 7.2. *Let $c \in \mathbb{C}^\times$ be such that $c^2 \notin q^\mathbb{Z}$. If $\mathbf{n}_{i,a}^- \mathbf{m} \in \text{wt}_\ell(N_{i,a}^-)$, then $\mathbf{n}_{c,a}^{(i)} \mathbf{m} \in \text{wt}_\ell(L(\mathbf{n}_{c,a}^{(i)}))$ and $\dim(N_{i,a}^- \mathbf{n}_{i,a}^- \mathbf{m}) \leq \dim L(\mathbf{n}_{c,a}^{(i)})_{\mathbf{n}_{c,a}^{(i)} \mathbf{m}}$.*

Proof. Combining Definition 3.1 (i) with Definition 5.2, we have

$$\mathbf{n}_{i,a}^- \equiv \mathbf{n}_{c,a}^{(i)} \prod_{j \sim i} \Psi_{j,aq_{ij}c_{ij}^2}^{-1}.$$

Viewing $N_{i,a}^-$ as a sub-quotient of $L(\mathbf{n}_{c,a}^{(i)}) \otimes (\otimes_{j \sim i} L_{j,aq_{ij}c_{ij}^2}^-) \otimes D$ with D being a one-dimensional $Y_q(\mathfrak{g})$ -module, we have $\mathbf{m} = \mathbf{m}' \prod_{j \sim i} \mathbf{m}^j$ with

$$\mathbf{n}_{c,a}^{(i)} \mathbf{m}' \in \text{wt}_\ell(L(\mathbf{n}_{c,a}^{(i)})), \quad \Psi_{j,aq_{ij}c_{ij}^2}^{-1} \mathbf{m}^j \in \text{wt}_\ell(L_{j,aq_{ij}c_{ij}^2}^-) \quad \text{for } j \sim i.$$

By Corollary 5.1 and Lemmas 4.1 and 4.2 we have:

- (1) $\mathbf{m}, \mathbf{m}' \in \widehat{\mathcal{Q}}^- q^{\mathbf{Q}^-}$ and \mathbf{m} is a monomial in the $A_{i',b}^{-1}$ with $i' \in I_0$ and $b \in aq^\mathbb{Z}$;
- (2) \mathbf{m}^j is a monomial in the $A_{i',b'}^{-1}$ with $i' \in I_0$ and $b' \in \{ac^2, ac^{-2}\}q^\mathbb{Z}$ for $j \sim i$.

Since $\{ac^2, ac^{-2}\}q^\mathbb{Z}$ and $aq^\mathbb{Z}$ do not intersect, $\mathbf{m}^j = 1$ and $\mathbf{m}' = \mathbf{m}$. \square

For $m_1, m_2 \in \mathbb{Z}_{>0}$ with $m_1 < m_2$, let $Z_{i,a}^{m_1, m_2}$ be the irreducible $U_q(\widehat{\mathfrak{g}})$ -module of highest ℓ -weight $\mathbf{n}_{q^{m_2}, a}^{(i)} (\mathbf{n}_{q^{m_1}, a}^{(i)})^{-1} = \prod_{j \sim i} \omega_{q_{ij}^{m_1 - m_2}, aq_{ij}^{1+2m_1}}^{(j)}$; by Lemma 1.5 (4) it is finite-dimensional. Fix $v^{m_1, m_2} \in Z_{i,a}^{m_1, m_2}$ to be a highest ℓ -weight vector.

Lemma 7.3. *The $U_q(\widehat{\mathfrak{g}})$ -module $N_{m_1, a}^{(i)} \otimes Z_{i, a}^{m_1, m_2} \otimes Z_{i, a}^{m_2, m_3}$ is of highest ℓ -weight for $0 < m_1 < m_2 < m_3$.*

Proof. We shall assume $1 \leq i \leq M$. The case $M+1 < i < M+N$ can be deduced from $1 \leq i < M$ using \mathbb{D} . (See typical arguments in the proof of Lemma 8.2.)

Suppose $1 \leq i < M$. By Corollary 6.10, $Z_{i, a}^{m_1, m_2} \simeq \otimes_{j \sim i} W_{m_2 - m_1, aq^{-2m_1 - 2}}^{(j)}$. The tensor product $W_{1, aq}^{(i)} \otimes (\otimes_{j \sim i} W_{m_1, aq^{-2}}^{(j)})$ satisfies Condition (6.33) and is of highest ℓ -weight. Its irreducible quotient is $\simeq N_{m_1, a}^{(i)}$. Next,

$$W_{1, aq}^{(i)} \otimes (\otimes_{j \sim i} W_{m_1, aq^{-2}}^{(j)}) \otimes (\otimes_{j \sim i} W_{m_2 - m_1, aq^{-2m_1 - 2}}^{(j)}) \otimes (\otimes_{j \sim i} W_{m_3 - m_2, aq^{-2m_2 - 2}}^{(j)})$$

also satisfies Condition (6.33), and is of highest ℓ -weight, implying that $N_{m_1, a}^{(i)} \otimes Z_{i, a}^{m_1, m_2} \otimes Z_{i, a}^{m_2, m_3}$ is of highest ℓ -weight.

Suppose $i = M$. Consider the tensor product of fundamental modules:

$$T := V_{N, aq^{-N-3}}^- \otimes (\otimes_{l=1}^{m_3} V_{N-1, aq^{-N+2l-1}}^-) \otimes (\otimes_{k=1}^{m_3} V_{M-1, aq^{-N+2M-2k-1}}^+).$$

By Lemma 6.11, T is of highest ℓ -weight and

$$\begin{aligned} T &\cong V_{N, aq^{-N-3}}^- \otimes (\otimes_{l=1}^{m_1} V_{N-1, aq^{-N+2l-1}}^-) \otimes (\otimes_{k=1}^{m_1} V_{M-1, aq^{-N+2M-2k-1}}^+) \otimes \\ &\quad (\otimes_{l=m_1+1}^{m_2} V_{N-1, aq^{-N+2l-1}}^-) \otimes (\otimes_{k=m_1+1}^{m_2} V_{M-1, aq^{-N+2M-2k-1}}^+) \\ &\quad (\otimes_{l=m_2+1}^{m_3} V_{N-1, aq^{-N+2l-1}}^-) \otimes (\otimes_{k=m_2+1}^{m_3} V_{M-1, aq^{-N+2M-2k-1}}^+). \end{aligned}$$

Let T_1, T_2, T_3 denote the above tensor products of the first, second, and third row at the right-hand side. They are of highest ℓ -weight. By Equation (6.34),

$$V_{N, aq^{-N-3}}^- \simeq W_{1, aq^{-1}}^{(M-)}, \quad V_{N-1, aq^{-N+1}}^- \simeq W_{1, aq^2}^{(M+1)}, \quad V_{M-1, aq^{-N+2M-3}}^- \simeq W_{1, aq^{-2}}^{(M-1)}.$$

By Definition 5.2, the irreducible quotients of T_1, T_2, T_3 are $\simeq N_{m_1, a}^{(M)}, Z_{M, a}^{m_1, m_2}$ and $Z_{M, a}^{m_2, m_3}$, proving the cyclicity statement. \square

Let $0 < m_1 < m_2$. The tensor product $N_{m_1,a}^{(i)} \otimes Z_{i,a}^{m_1,m_2}$ being of highest ℓ -weight, its irreducible quotient is isomorphic to $N_{m_2,a}^{(i)}$. There exists a unique morphism of $U_q(\widehat{\mathfrak{g}})$ -modules $\mathcal{F}_{m_2,m_1} : N_{m_1,a}^{(i)} \otimes Z_{i,a}^{m_1,m_2} \longrightarrow N_{m_2,a}^{(i)}$ which sends $v^{m_1} \otimes v^{m_1,m_2}$ to v^{m_2} . As in [32, §4.2], define

$$F_{m_2,m_1} : N_{m_1,a}^{(i)} \longrightarrow N_{m_2,a}^{(i)}, \quad w \mapsto \mathcal{F}_{m_2,m_1}(w \otimes v^{m_1,m_2}).$$

Then $(\{N_{m,a}^{(i)}\}, \{F_{m_2,m_1}\})$ constitutes an inductive system of vector superspaces: $F_{m_3,m_2}F_{m_2,m_1} = F_{m_3,m_1}$ for $0 < m_1 < m_2 < m_3$. The proof is the same as that of [51, Proposition 4.1 (3)], based on Lemma 7.3.

Lemma 7.4. *Let $0 < m_1 < m_2$. We have $F_{m_2,m_1}x_{j,n}^+ = x_{j,n}^+F_{m_2,m_1}$ for $j \in I_0$ and $n \in \mathbb{Z}$. The linear map F_{m_2,m_1} is injective.*

Proof. For the first statement, note that in Equation (1.10) the summation annihilates $N_{m_1,a}^{(i)} \otimes v^{m_1,m_2}$ and can be omitted in computing F_{m_2,m_1} from the $U_q(\widehat{\mathfrak{g}})$ -linear map \mathcal{F}_{m_2,m_1} . Let $K \subseteq N_{m_1,a}^{(i)}$ be the kernel of F_{m_2,m_1} and assume $K \neq 0$. Then K is stable by the $x_{j,n}^+$. Since $N_{m_1,a}^{(i)}$ is finite-dimensional, there exists a non-zero vector $w \in K$ killed by the $x_{j,n}^+$. Such w must be proportional to the highest ℓ -weight vector v^{m_1} as $N_{m_1,a}^{(i)}$ is irreducible. This is absurd because $F_{m_2,m_1}(v^{m_1}) = v^{m_2} \neq 0$. \square

Lemma 7.5. *For $j \in I_0$ and $m_2 > m_1 > 0$ we have*

$$\begin{aligned} \phi_j^\pm(z)F_{m_2,m_1} &= F_{m_2,m_1}\phi_j^\pm(z) \quad \text{if } |j-i| \neq 1, \\ \phi_j^\pm(z)F_{m_2,m_1} &= q_{ij}^{m_1-m_2} \frac{1 - zaq_{ij}^{1+2m_2}}{1 - zaq_{ij}^{1+2m_1}} \times F_{m_2,m_1}\phi_j^\pm(z) \quad \text{if } |j-i| = 1, \\ K_M^\pm(z)F_{m_2,m_1} &= (q^{m_2}A_{m_1}^\pm(z) + B_{m_1}^\pm(z) + q^{-m_2}C_{m_1}^\pm(z)) \times F_{m_2,m_1}K_M^\pm(z). \end{aligned}$$

Here $A_{m_1}^\pm(z), B_{m_1}^\pm(z), C_{m_1}^\pm(z) \in \mathbb{C}[[z^{\pm 1}]]$.

Proof. The arguments in the proof of [32, Proposition 4.2] work here, because of Equation (1.9). The extra power series in $z^{\pm 1}$ at the second and third rows are eigenvalues of $\phi_j^\pm(z), K_M^\pm(z)$ associated with v^{m_1,m_2} . \square

Lemma 7.6. *For $j \in I_0$ and $m_2 - 1 > m > 0$ we have*

$$\begin{aligned} x_{j,0}^-F_{m_2,m} &= F_{m_2,m}x_{j,0}^- \quad \text{if } |j-i| \neq 1, \\ x_{j,0}^-F_{m_2,m} &= F_{m_2,m+1}(q^{m_2}A_{j,m} + q^{-m_2}C_{j,m}) \quad \text{if } |j-i| = 1. \end{aligned}$$

Here $A_{j,m}, C_{j,m} : N_{m,a}^{(i)} \longrightarrow N_{m+1,a}^{(i)}$ are linear maps of parity $|\alpha_j|$.

Proof. By Lemma 7.3, the $U_q(\widehat{\mathfrak{g}})$ -module $Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ is of highest ℓ -weight with irreducible quotient $Z_{i,a}^{m,m_2}$; let $\mathcal{G}_{m_2,m}$ be the quotient map sending $v^{m,m+1} \otimes v^{m+1,m_2}$ to v^{m,m_2} . We claim that for $v \in N_{m,a}^{(i)}$, $v' \in Z_{i,a}^{m,m+1}$ and $j \in I_0$:

- (i) $\mathcal{F}_{m_2,m}(v \otimes \mathcal{G}_{m_2,m}(v' \otimes v^{m+1,m_2})) = F_{m_2,m+1}\mathcal{F}_{m+1,m}(v \otimes v')$;
- (ii) $x_{j,0}^-v^{m,m_2} = [(m_2 - m)\delta_{|j-i|,1}]_q \times \mathcal{G}_{m_2,m}(x_{j,0}^-v^{m,m+1} \otimes v^{m+1,m_2})$.

Here $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ for $n \in \mathbb{Z}$. Assume the claim for the moment. For $v \in N_{m,a}^{(i)}$, based on $\Delta(x_{j,0}^-) = 1 \otimes x_{j,0}^- + x_{j,0}^- \otimes \phi_{j,0}^-$ we compute $x_{j,0}^-F_{m_2,m}(v)$

$$\begin{aligned} &= x_{j,0}^-\mathcal{F}_{m_2,m}(v \otimes v^{m,m_2}) = \mathcal{F}_{m_2,m}\Delta(x_{j,0}^-)(v \otimes v^{m,m_2}) \\ &= \mathcal{F}_{m_2,m}(x_{j,0}^-v \otimes \phi_{j,0}^-v^{m,m_2}) + (-1)^{|v||\alpha_j|}\mathcal{F}_{m_2,m}(v \otimes x_{j,0}^-v^{m,m_2}) \\ &= q_{ij}^{(m_2-m)\delta_{|j-i|,1}}F_{m_2,m}(x_{j,0}^-v) + \\ &\quad (-1)^{|v||\alpha_j|}[(m_2 - m)\delta_{|j-i|,1}]_q\mathcal{F}_{m_2,m}(v \otimes \mathcal{G}_{m_2,m}(x_{j,0}^-v^{m,m+1} \otimes v^{m+1,m_2})) \end{aligned}$$

$$\begin{aligned}
&= q_{ij}^{(m_2-m)\delta_{|j-i|,1}} F_{m_2,m}(x_{j,0}^- v) + \\
&\quad (-1)^{|v||\alpha_j|} [(m_2-m)\delta_{|j-i|,1}]_q F_{m_2,m+1} \mathcal{F}_{m+1,m}(v \otimes x_{j,0}^- v^{m,m+1}),
\end{aligned}$$

which proves the lemma. The third and fourth identities used (ii) and (i).

Note that $\mathcal{F}_{m_2,m}(1_{N_{m,a}^{(i)}} \otimes \mathcal{G}_{m_2,m})$ and $\mathcal{F}_{m_2,m+1}(\mathcal{F}_{m+1,m} \otimes 1_{Z_{i,a}^{m+1,m_2}})$, as $U_q(\widehat{\mathfrak{g}})$ -linear maps from the highest ℓ -weight module $N_{m,a}^{(i)} \otimes Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ to $N_{m_2,a}^{(i)}$, are identical because they both send the highest ℓ -weight vector $v^m \otimes v^{m,m+1} \otimes v^{m+1,m_2}$ to v^{m_2} . Applying them to $v \otimes v' \otimes v^{m+1,m_2}$ gives (i).

From the proof of Lemma 7.3 it follows that $Z_{i,a}^{m,m_2}$ is \simeq irreducible quotient of a tensor product of KR modules associated to $j' \in I_0$ with $j' \sim i$. Let μ be the weight of v^{m,m_2} . If $|j-i| \neq 1$, then by Lemma 3.5, $\mu q^{-\alpha_j} \notin \text{wt}(Z_{i,a}^{m,m_2})$ and $x_{j,0}^- v^{m,m_2} = 0$. Suppose $j \sim i$. Then $(Z_{i,a}^{m,m_2})_{\mu q^{-\alpha_j}} = \mathbb{C} x_{j,0}^- v^{m,m_2}$ and $q_{ij} = q^{\pm 1}$. The equation $x_{j,0}^- \mathcal{G}_{m_2,m} = \mathcal{G}_{m_2,m} x_{j,0}^-$ applied to $v^{m,m+1} \otimes v^{m+1,m_2}$ gives

$$x_{j,0}^- v^{m,m_2} = \mathcal{G}_{m_2,m}(q_{ij}^{m_2-m-1} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} + v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}).$$

Consider the following vector in $Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ of weight $\mu q^{-\alpha_j}$:

$$w := q_{ij}^{-1} \frac{q_{ij}^{m+1-m_2} - q_{ij}^{m_2-m-1}}{q_{ij}^{-1} - q_{ij}} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} - v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}.$$

We have $\mathcal{G}_{m_2,m}(w) \in \mathbb{C} x_{j,0}^- v^{m,m_2}$ and $x_{j,0}^+ w = 0$. So $x_{j,0}^+ \mathcal{G}_{m_2,m}(w) = 0$. Now $x_{j,0}^+ x_{j,0}^- v^{m,m_2} \neq 0$ forces $\mathcal{G}_{m_2,m}(w) = 0$. We express $x_{j,0}^- v^{m,m_2}$

$$\begin{aligned}
&= \mathcal{G}_{m_2,m}(q_{ij}^{m_2-m-1} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} + v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}) + \mathcal{G}_{m_2,m}(w) \\
&= \left(q_{ij}^{m_2-m-1} + \frac{q_{ij}^{m-m_2} - q_{ij}^{m_2-m-2}}{q_{ij}^{-1} - q_{ij}} \right) \times \mathcal{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2}) \\
&= [m_2 - m]_{q_{ij}} \times \mathcal{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2}),
\end{aligned}$$

which proves (ii) because $[n]_{q_{ij}} = [n]_q$ for $n \in \mathbb{Z}$. \square

The above proof is different from that of [32, Proposition 4.5]: the polynomial behaviour with respect to q^{m_2} of the linear maps $x_{j,0}^- F_{m_2,m}$ is described in terms of $x_{j,0}^- v^{m,m_2}$. It can be used to simplify the proof of Proposition 5.3 in [51].

Proof of Proposition 5.4. For $r \in \mathbb{Z}_{\geq 0}$ let $K_{M,\pm r}^{\pm}$ be the coefficient of $z^{\pm r}$ in $K_M^{\pm}(z) \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$. As a superalgebra $U_q(\widehat{\mathfrak{g}})$ is generated by [49, Theorem 3.5]:

$$S := \{K_{M,\pm r}^{\pm}, \phi_{j,0}^{\pm}, x_{j,0}^{\pm}, x_{j,n}^{\pm} \mid r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}, j \in I_0\}.$$

By Lemmas 7.4–7.6, for $m \in \mathbb{Z}_{>0}$ and $s \in S$, there exists a Laurent polynomial $P_{m,s}(u) \in \text{Hom}_{\mathbb{C}}(N_{m,a}^{(i)}, N_{m+1,a}^{(i)})[u, u^{-1}]$ which has non-zero coefficients only at $u, 1, u^{-1}$, and $sF_{m_2,m} = F_{m_2,m+1}P_{m,s}(q^{m_2})$ for $m_2 > m+1$. Since q is not a root of unity, the injectivity of the F_{m_2,m_1} ensures the uniqueness of the $P_{m,s}(u)$.

The generic asymptotic construction of [51, §4] can be applied to the inductive system $(\{N_{m,a}^{(i)}\}, \{F_{m_2,m_1}\})$. Let $N_{\infty,a}^{(i)}$ be its inductive limit. For $c \in \mathbb{C}^{\times}$, there exists a unique representation ρ^c of $U_q(\widehat{\mathfrak{g}})$ on $N_{\infty,a}^{(i)}$ defined by:

$$\rho^c(s) := \lim_{m \rightarrow \infty} P_{m,s}(c) \in \text{Hom}_{\mathbb{C}}(N_{\infty,a}^{(i)}, N_{\infty,a}^{(i)}) \quad \text{for } s \in S.$$

Let $\mathcal{N}_{c,a}^{(i)}$ denote the resulting $U_q(\widehat{\mathfrak{g}})$ -module. It contains a unique highest ℓ -weight vector $v^{\infty} := \lim_{m \rightarrow \infty} v^m$ of ℓ -weight $\mathbf{n}_{c,a}^{(i)}$ by Lemma 7.4. Moreover, by Lemma 7.5,

$$\chi_q(\mathcal{N}_{c,a}^{(i)}) = \mathbf{n}_{c,a}^{(i)} \times \lim_{m \rightarrow \infty} \tilde{\chi}_q(N_{m,a}^{(i)}).$$

Let us prove $\lim_{m \rightarrow \infty} \tilde{\chi}_q(N_{m,a}^{(i)}) = \tilde{\chi}_q(N_{i,a}^-)$. By Lemma 7.1, the left-hand side is bounded by the right-hand side. Notice that $L(\mathbf{n}_{c,a}^{(i)})$ is a sub-quotient of $\mathcal{N}_{c,a}^{(i)}$ for $c \in \mathbb{C}^\times$. Suppose $c^2 \notin q^\mathbb{Z}$. By Lemma 7.2, $\tilde{\chi}_q(N_{i,a}^-)$ is bounded by $\tilde{\chi}_q(L(\mathbf{n}_{c,a}^{(i)}))$ and so by $(\mathbf{n}_{c,a}^{(i)})^{-1} \chi_q(\mathcal{N}_{c,a}^{(i)})$, which is exactly the left-hand side. This implies the desired identity and the irreducibility of $\mathcal{N}_{c,a}^{(i)}$ because $\chi_q(\mathcal{N}_{c,a}^{(i)}) = \chi_q(L(\mathbf{n}_{c,a}^{(i)}))$. \square

One can have asymptotic modules $\mathcal{M}_{c,a}^{(i)}$ over $U_q(\widehat{\mathfrak{g}})$ as limits of $M_{q^m,a}^{(i)}$ (as in [32, §7.2], which is slightly different from the limit construction of $\mathcal{N}_{c,a}^{(i)}$). Then Equation (5.30) holds with M replaced by \mathcal{M} for all $c, d \in \mathbb{C}^\times$.

Proposition 7.7. *The $U_q(\widehat{\mathfrak{g}})$ -module $\mathcal{W}_{c_1,a_1}^{(i_1)} \otimes \mathcal{W}_{c_2,a_2}^{(i_2)} \otimes \cdots \otimes \mathcal{W}_{c_s,a_s}^{(i_s)}$, with $i_l \in I_0$ and $c_l, a_l \in \mathbb{C}^\times$, is irreducible if $a_l c_l^{-2} \notin a_k q^\mathbb{Z}$ for all $1 \leq l, k \leq s$.*

Proof. Let $L := \otimes_{l=1}^s L_{i_l,a_l}^-$ and $S = L(\prod_{l=1}^s \omega_{c_l,a_l}^{(i_l)})$, viewed as irreducible $Y_q(\widehat{\mathfrak{g}})$ -modules by Corollary 4.3. S is a sub-quotient of the tensor product T in the proposition. Let ω, ω' be the highest ℓ -weights of L, S respectively. Then $\chi_q(T) = \omega' \tilde{\chi}_q(L)$ by Proposition 5.3. It suffices to prove that $\dim L_{\mathbf{n}\omega} \leq \dim S_{\mathbf{n}\omega'}$ for all $\mathbf{n}\omega \in \text{wt}_\ell(L)$. Viewing L as a sub-quotient of $S \otimes D$ where $D \simeq \otimes_{l=1}^s L_{i_l,a_l c_l^{-2}}^-$, we can adapt the proof of Lemma 7.2 to the present situation. \square

It follows that the tensor products of the \mathcal{W} at the right-hand side of Equations (5.29)–(5.30) are irreducible $U_q(\widehat{\mathfrak{g}})$ -modules for $c^2, d^2 \notin q^\mathbb{Z}$.

8. PROOF OF EXTENDED T-SYSTEMS: THEOREM 3.4

The idea is to provide lower and upper bounds for $\dim(D_{m,a}^{(i,s)})$. We recall from the proof of Corollary 3.6 that the $U_q(\widehat{\mathfrak{g}})$ -module $W_{m,aq_i^{2m+1}}^{(i)} \otimes W_{m+s,aq_i^{2m-1}}^{(i)}$ has at least two irreducible sub-quotients: $L(\varpi_{m+s+1,aq_i^{2m+1}}^{(i)} \varpi_{m-1,aq_i^{2m-1}}^{(i)})$ and $D_{m,a}^{(i,s)}$.

Lemma 8.1. *For $i \in I_0 \setminus \{M\}$, the $U_q(\widehat{\mathfrak{g}})$ -module $W_{m+s,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s)}$ has at least two sub-quotients: $L(\mathbf{d}_{m,a}^{(i,s-1)} \varpi_{m+s+1,aq_i^{2m+1}}^{(i)})$ and $L(\mathbf{d}_{m,a}^{(i,0)} \varpi_{m,aq_i^{2m+1}}^{(i)})$.*

Proof. Set $T := W_{m+s,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s)}$ and $S := L(\mathbf{d}_{m,a}^{(i,s-1)} \varpi_{m+s+1,aq_i^{2m+1}}^{(i)})$. By Definition 3.1 (ii), S is an irreducible sub-quotient of T . By Corollary 3.6,

$$\mathbf{m}' := \mathbf{m} \prod_{l=1}^s A_{i,aq_i^{2-2l}}^{-1} = \mathbf{d}_{m+s,aq_i^{-2s}}^{(i,0)} \varpi_{m,aq_i^{2m+1}}^{(i)} \in \text{wt}_\ell(T).$$

Viewing S as an irreducible sub-quotient of $W_{m+s+1,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s-1)}$ and using Lemma 3.5 and Corollary 3.6, we have $\mathbf{m}' \notin \text{wt}_\ell(S)$. Let $\mu := (3m+2s)\varpi_i - m\alpha_i$ so that $\varpi(\mathbf{m}) = q^\mu$ and $\varpi(\mathbf{m}') = q^{\mu-s\alpha_i}$. Then $\dim T_{q^{\mu-t\alpha_i}} = t+1$ for $0 \leq t \leq s$.

Let $v_0 \in S$ be a highest ℓ -weight vector and let U_i be the subalgebra in the proof of Corollary 3.6. Then $U_i v_0$ is an irreducible $U_{q_i}(\widehat{\mathfrak{sl}}_2)$ -module of highest ℓ -weight

$$\mathbf{m}_i := (Y_{aq_i^{-1}} Y_{i,aq_i^{-3}} \cdots Y_{i,aq_i^{1-2s}}) (Y_{i,aq_i^{2m+1}} Y_{i,aq_i^{2m-1}} \cdots Y_{i,aq_i^{3-2s}})$$

and factorizes as $L^i(Y_{aq_i^{-1}} Y_{i,aq_i^{-3}} \cdots Y_{i,aq_i^{3-2s}}) \otimes L^i(Y_{i,aq_i^{2m+1}} Y_{i,aq_i^{2m-1}} \cdots Y_{i,aq_i^{1-2s}})$; if $s = 1$ then the first tensor factor is trivial. For $1 \leq t \leq s$, the weight space $S_{q^{\mu-t\alpha_i}}$ is spanned by the $x_{i,n_1}^- x_{i,n_2}^- \cdots x_{i,n_t}^- v_0 \in U_i v_0$ with $n_l \in \mathbb{Z}$ for $1 \leq l \leq t$ and is therefore of dimension $\min(s, t+1)$. Since $\mathbf{m}_i \prod_{l=1}^s (Y_{aq_i^{1-2l}} Y_{aq_i^{3-2l}})^{-1}$ is not an ℓ -weight of $L^i(\mathbf{m}_i)$, we must have $\mathbf{m}' \notin \text{wt}_\ell(S)$, as in the proof of Corollary 3.6.

It follows that $\chi_q(T) - \chi_q(S)$ is \mathbf{m}' plus terms of the form $\mathbf{m}'' \in \mathbf{R}$ with $\varpi(\mathbf{m}'') \notin \varpi(\mathbf{m}')q^{\mathbb{Q}^+}$, forcing $L(\mathbf{m}')$ to be an irreducible sub-quotient of T . \square

Lemma 8.2. *Let $i \in I_0 \setminus \{M\}$. The $U_q(\widehat{\mathfrak{g}})$ -modules $W_{m,aq_i^{2m+1}}^{(i)} \otimes W_{m+s,aq_i^{2m-1}}^{(i)}$ and $W_{m+s,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s)}$ are of highest ℓ -weight, while $W_{m+s+1,aq_i^{2m+1}}^{(i)} \otimes W_{m-1,aq_i^{2m-1}}^{(i)}$, $D_{m+s,aq_i^{-2s}}^{(i,0)} \otimes W_{m,aq_i^{2m+1}}^{(i)}$ and $W_{m+s+1,aq_i^{2m+1}}^{(i)} \otimes D_{m,a}^{(i,s-1)}$ are irreducible.*

Proof. Assume $i < M$. Notice that $T_{m,a}^{(i,s)} := W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)} \otimes W_{s,aq^{-1}}^{(i)}$ satisfies Condition (6.33) and is of highest ℓ -weight. By Remark 3.2, the irreducible quotient of $T_{m,a}^{(i,s)}$ is $\simeq D_{m,a}^{(i,s)}$. To prove that the five tensor products in the lemma are of highest ℓ -weight, we can replace D by T and show that the resulting tensor products of KR modules satisfy Condition (6.33). For example the last tensor product corresponds to $W_{m+s+1,aq^{2m+1}}^{(i)} \otimes W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)} \otimes W_{s-1,aq^{-1}}^{(i)}$.

Next, $S_{m,a}^{(i,s)} := W_{s,a^{-1}q^{2s+1}}^{(i)} \otimes W_{m,a^{-1}}^{(i-1)} \otimes W_{m,a^{-1}}^{(i+1)}$ also satisfies Condition (6.33) and is of highest ℓ -weight, the irreducible quotient of which is $\simeq (T_{m,a}^{(i,s)})^\vee$. To establish the irreducibility of the last three tensor products in the lemma, we take twisted duals as in Lemma 6.9, replace D^\vee by S , and check Condition (6.33) for the resulting tensor products of KR modules. Take the fourth as an example: $W_{m+s,a^{-1}q^{2s}}^{(i-1)} \otimes W_{m+s,a^{-1}q^{2s}}^{(i+1)} \otimes W_{m,a^{-1}q^{-1}}^{(i)}$ is of highest ℓ -weight.

This proves the lemma in the case $i < M$.

Assume $i > M$. By Lemma 1.8, $\mathbb{D}(W_{m,a}^{(i)}) \simeq W_{m,aq^{N-M-2+2m}}^{(M+N-i)}$ as $U_q(\widehat{\mathfrak{g}}')$ -modules. Applying \mathbb{D}^{-1} to the $U_q(\widehat{\mathfrak{g}}')$ -modules $T_{m,a}^{(M+N-i,s)}$, $S_{m,a}^{(M+N-i,s)}$ we obtain that $D_{m,a}^{(i,s)}$ and $(D_{m,a}^{(i,s)})^\vee$ are \simeq the irreducible quotients of the highest ℓ -weight modules

$$W_{m,aq^{-2m}}^{(i+1)} \otimes W_{m,aq^{-2m}}^{(i-1)*} \otimes W_{s,aq}^{(i)}, \quad W_{s,a^{-1}q^{3-2s}}^{(i)} \otimes W_{m,a^{-1}q^4}^{(i-1)*} \otimes W_{m,a^{-1}q^4}^{(i+1)}$$

respectively. Here $W_{m,a}^{(M)*} := W_{m,a}^{(M-)}$ and $W_{m,a}^{(j)*} = W_{m,a}^{(j)}$ for $j > M$. By replacing D, D^\vee with these tensor products, we obtain eight tensor products of KR modules $W_{m,b}^{(j)}, W_{m,b}^{(M-)}$ with $j > M$ and need to show that they are of highest ℓ -weight. Applying \mathbb{D} gives tensor products of KR modules $W_{m,b}^{(j)}$ with $j \leq M$ over $U_q(\widehat{\mathfrak{g}}')$, which are shown to satisfy Condition (6.31). Consider the last tensor product in the lemma as an example. Let us prove that the $U_q(\widehat{\mathfrak{g}})$ -modules

$$\begin{aligned} T_1 &:= W_{m+s+1,aq^{-2m-1}}^{(i)} \otimes W_{m,aq^{-2m}}^{(i+1)} \otimes W_{m,aq^{-2m}}^{(i-1)*} \otimes W_{s-1,aq}^{(i)}, \\ T_2 &:= W_{m+s+1,a^{-1}q^{3-2s}}^{(i)} \otimes W_{s-1,a^{-1}q^{5-2s}}^{(i)} \otimes W_{m,a^{-1}q^4}^{(i-1)*} \otimes W_{m,a^{-1}q^4}^{(i+1)} \end{aligned}$$

are of highest ℓ -weight. Applying \mathbb{D} to T_1, T_2 give $(c = q^{N-M-2}, j = M + N - i)$:

$$\begin{aligned} T_1' &= W_{s-1,acq^{2s-1}}^{(j)} \otimes W_{m,ac}^{(j+1)} \otimes W_{m,ac}^{(j-1)} \otimes W_{m+s+1,ac^{2s+1}}^{(j)}, \\ T_2' &= W_{m,a^{-1}cq^{2m+4}}^{(j-1)} \otimes W_{m,a^{-1}cq^{2m+4}}^{(j+1)} \otimes W_{s-1,a^{-1}cq^3}^{(j)} \otimes W_{m+s+1,a^{-1}cq^{2m+5}}^{(j)}. \end{aligned}$$

The $U_q(\widehat{\mathfrak{g}}')$ -modules T_1', T_2' satisfy Condition (6.31). \square

For $i \in I_0$ and $m \in \mathbb{Z}_{>0}$ let $d_m^{(i)} := \dim(W_{m,a}^{(i)})$; it is independent of $a \in \mathbb{C}^\times$ because $\Phi_a^*(W_{m,1}^{(i)}) \cong W_{m,a}^{(i)}$ by Equation (1.1).

Theorem 8.3. [42] $(d_m^{(i)})^2 = d_{m+1}^{(i)} d_{m-1}^{(i)} + d_m^{(i-1)} d_m^{(i+1)}$ for $1 \leq i < M$.

Proof. For $\mu \in \mathcal{P}$, up to normalization $\mathcal{T}_{\emptyset \subset \mu}(u)$ in [42, (2.15)] can be identified with $\chi_q(V_q^-(\mu; a))$ in Equation (2.19). The dimension identity is a consequence of [42, (3.2)], which in turn comes from Jacobi identity of determinants. \square

Proof of Theorem 3.4. By Lemma 8.2, the surjective morphisms of $U_q(\widehat{\mathfrak{g}})$ -modules in Theorem 3.4 exist (because the third terms are irreducible quotients

of the second terms) and their kernels admit irreducible sub-quotients $D_{m,a}^{(i,s)}$ and $D_{m+s,aq_i^{-2s}}^{(i,0)} \otimes W_{m,aq_i^{2m+1}}^{(i)}$ respectively. This gives:

- (1) $\dim(D_{m,a}^{(i,s)}) \leq d_m^{(i)} d_{m+s}^{(i)} - d_{m+s+1}^{(i)} d_{m-1}^{(i)}$;
- (2) $\dim(D_{m+s,aq_i^{-2s}}^{(i,0)}) d_m^{(i)} \leq d_{m+s}^{(i)} \dim(D_{m,a}^{(i,s)}) - d_{m+s+1}^{(i)} \dim(D_{m,a}^{(i,s-1)})$.

We prove the equality in (1)–(2) by induction on s . Suppose $s = 0$; (2) is trivial. If $i < M$, then by Definition 3.1 (ii) and Corollary 6.10,

$$D_{m,a}^{(i,0)} \simeq W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)}.$$

This together with Theorem 8.3 shows that equality holds in (1). Making use of \mathbb{D} , we can remove the assumption $i < M$, as in the proof of Lemma 8.2.

Suppose $s > 0$. In (2) the induction hypothesis applied to $0, s-1$ indicates that

$$\begin{aligned} ((d_{m+s}^{(i)})^2 - d_{m+s+1}^{(i)} d_{m+s-1}^{(i)}) d_m^{(i)} &\leq d_{m+s}^{(i)} \dim(D_{m,a}^{(i,s)}) \\ &\quad - d_{m+s+1}^{(i)} (d_m^{(i)} d_{m+s-1}^{(i)} - d_{m+s}^{(i)} d_{m-1}^{(i)}); \end{aligned}$$

namely, $\dim(D_{m,a}^{(i,s)}) \geq d_m^{(i)} d_{m+s}^{(i)} - d_{m+s+1}^{(i)} d_{m-1}^{(i)}$. This implies that in (1), and henceforth in the above inequality and in (2), \leq can be replaced by $=$. \square

Remark 8.4. Let $1 \leq i < M$. Apply \mathbb{D}^{-1} to the second exact sequence in category \mathcal{O}' of Theorem 3.4 involving $D_{m,a}'^{(M+N-i,1)}$ and take normalized q -characters:

$$\begin{aligned} \tilde{\chi}_q(N_{m,a}^{(i)}) \tilde{\chi}_q(W_{m+1,aq^{-1}}^{(i)}) &= \tilde{\chi}_q(W_{m+2,aq}^{(i)}) \prod_{j \in I_0: j \sim i} \tilde{\chi}_q(W_{m,aq^{-2}}^{(j)}) \\ &\quad + A_{i,a}^{-1} \times \tilde{\chi}_q(W_{m,aq^{-3}}^{(i)}) \prod_{j \in I_0: j \sim i} \tilde{\chi}_q(W_{m+1,a}^{(j)}). \end{aligned}$$

Setting $m \rightarrow \infty$ recovers the normalized q -characters of Equation (5.28). The second exact sequence of Theorem 3.4 is likely to be true for $i = M$.

Theorem 3.4 together with its proof could be adapted to quantum affine algebras, in view of the cyclicity results of [11] and T-system [40, 29]. The second and third terms of the first exact sequence appeared in the proof of [22, Theorem 4.1] as V', V by setting $(a, m, s) = (q_i^{-3}, m_2 + 1, m_1 - m_2 - 2)$. In the context of graded representations of current algebras [15, Theorem 2] by taking $(\ell, \lambda) = (m + s, m\omega_i)$ so that $\nu = (2m + s)\omega_i - m\alpha_i$, the exact sequence therein is an injective resolution of the Demazure module $D(\ell, \nu)$ by fusion products of KR modules. It is natural to expect that $D_{m,1}^{(i,s)}$ admits a classical limit ($q = 1$) as $D(\ell, \nu)$; this is true when $m = s = 1$, as a particular case of [10, Theorem 1].

9. TRANSFER MATRICES AND BAXTER OPERATORS

Let us fix an integer $\ell \in \mathbb{Z}_{>0}$ (length of spin chain) and complex numbers $b_j \in \mathbb{C}^\times \setminus q^\mathbb{Z}$ for $1 \leq j \leq \ell$ (inhomogeneity parameters). We shall construct an action of $K_0(\mathcal{O})$ on the vector superspace $\mathbf{V}^{\otimes \ell}$ as in [21, §5]. This is the XXZ spin chain with twisted periodic boundary condition, with $\mathbf{V}^{\otimes \ell}$ referred to as the quantum space and $W \in \mathcal{O}$ an auxiliary space.

Following Definition 1.4, let \mathcal{E} be the subset of \mathcal{E}_ℓ consisting of the $\sum_{\hat{p}} n_{\hat{p}} \hat{p} \in \mathcal{E}_\ell$ with $\hat{p} \in \mathfrak{P}$ if $n_{\hat{p}} \neq 0$. Note that \mathcal{E} is a sub-ring and $\chi(W) \in \mathcal{E}$ for $W \in \mathcal{O}$.

We identify $\underline{i} = i_1 i_2 \cdots i_\ell \in I^\ell$, an I -string of length ℓ , with the basis vector $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_\ell}$ of $\mathbf{V}^{\otimes \ell}$. Let $E_{\underline{i}\underline{j}} \in \text{End}(\mathbf{V}^{\otimes \ell})$ be the elementary matrix $\underline{k} \mapsto \delta_{\underline{j}\underline{k}} \underline{i}$ for $\underline{i}, \underline{j} \in I^\ell$, and let $\epsilon_{\underline{i}} := \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_\ell} \in \mathbf{P}$.

To a $Y_q(\mathfrak{g})$ -module W in category \mathcal{O} is by definition attached an matrix $S^W(z)$, a power series in z with values in $\text{End}(W) \otimes \text{End}(\mathbf{V})$. We decompose

$$S_{1,\ell+1}^W(zb_\ell) \cdots S_{13}^W(zb_2) S_{12}^W(zb_1) = \sum_{\underline{i}, \underline{j} \in I^\ell} S_{\underline{i}\underline{j}}^W(z) \otimes E_{\underline{i}\underline{j}} \in \text{End}(W) \otimes \text{End}(\mathbf{V})^{\otimes \ell}[[z]].$$

Then $S_{\underline{i}\underline{j}}^W(z) = \pm s_{i_\ell j_\ell}^W(zb_\ell) \cdots s_{i_2 j_2}^W(zb_2) s_{i_1 j_1}^W(zb_1)$ and it sends one weight space W_p for $p \in \mathfrak{P}$ to another of weight $pq^{\epsilon_{\underline{i}} - \epsilon_{\underline{j}}}$. Its trace over W_p is well-defined: either 0 if $\epsilon_{\underline{i}} \neq \epsilon_{\underline{j}}$; or the usual non-graded trace of $S_{\underline{i}\underline{j}}^W(z)|_{W_p} \in \text{End}(W_p)$ if $\epsilon_{\underline{i}} = \epsilon_{\underline{j}}$.

Definition 9.1. Let $W \in \mathcal{O}$. Its associated *transfer matrix* is

$$t_W(z) := \sum_{\underline{i}, \underline{j} \in I^\ell} \left(\sum_{p \in \text{wt}(W)} p \times \text{Tr}_{W_p}(S_{\underline{i}\underline{j}}^W(z)) \right) E_{\underline{i}\underline{j}},$$

viewed as a power series in z with values in $\text{End}(\mathbf{V}^{\otimes \ell}) \otimes_{\mathbb{Z}} \mathcal{E}$.

In [6, 44] (for $U_q(\widehat{\mathfrak{g}})$) and [23] (for an arbitrary non-twisted quantum affine algebra), transfer matrices are partial traces of universal R-matrices $\mathcal{R}(z)$. Since the existence of $\mathcal{R}(z)$ for $U_q(\widehat{\mathfrak{g}})$ is not clear to the author (except the simplest case $\mathfrak{gl}(1|1)$ in [48]), we use a different transfer matrix based on RTT. One should imagine $S^W(z)$ as the specialization of $\mathcal{R}(z)$ at $W \otimes \mathbf{V}$.

Example 9.2. Consider the one-dimensional module $\mathbb{C}^{\widehat{p}}$ in Example 1.3:

$$t_{\mathbb{C}^{\widehat{p}}}(z)\underline{i} = \varpi(\widehat{p}) \times \underline{i} \prod_{l=1}^{\ell} p_{i_l}(zb_l) \quad \text{for } \underline{i} \in I^\ell.$$

Proposition 9.3. Let $X, Y \in \mathcal{O}$ and $a \in \mathbb{C}^\times$. The following equations hold:

$$t_{\Phi_a^* X}(z) = t_X(za), \quad t_X(z)t_Y(z) = t_{X \otimes Y}(z), \quad t_X(z)t_Y(w) = t_Y(w)t_X(z).$$

Proof. We mainly prove the second equation; the first one is almost clear from Definition 9.1 and Equation (1.1), and the third one in the same way as [23, Theorem 5.3] based on the commutativity of $K_0(\mathcal{O})$. For $\underline{i}, \underline{j} \in I^\ell$:

$$\begin{aligned} S_{\underline{i}\underline{j}}^{X \otimes Y}(z) \otimes E_{\underline{i}\underline{j}} &= \prod_{r=\ell}^1 s_{i_r j_r}^{X \otimes Y}(zb_r) \otimes E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_\ell j_\ell} \\ &= \sum_{\underline{k} \in I^\ell} \prod_{r=\ell}^1 \left((-1)^{|E_{i_r k_r}| |E_{k_r j_r}|} s_{i_r k_r}^X(zb_r) \otimes s_{k_r j_r}^Y(zb_r) \right) \otimes E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_\ell j_\ell} \\ &= \sum_{\underline{k} \in I^\ell} (S_{\underline{i}\underline{k}}^X(z) \otimes 1 \otimes E_{\underline{i}\underline{k}}) (1 \otimes S_{\underline{k}\underline{j}}^Y(z) \otimes E_{\underline{k}\underline{j}}). \end{aligned}$$

After taking trace over $X_p \otimes Y_{p'}$, only the terms with $\epsilon_{\underline{i}} = \epsilon_{\underline{k}} = \epsilon_{\underline{j}}$ survive and so all the tensor components are of even parity, implying the second equation. \square

Let $\varphi : \mathfrak{P} \rightarrow \mathbb{C}^\times$ be a morphism of multiplicative groups (typical examples are $((p_i)_{i \in I}; s) \mapsto (-1)^s$ and $((p_i)_{i \in I}; s) \mapsto (-1)^s \times \prod_{i \in I} p_i$). If W is a finite-dimensional $Y_q(\mathfrak{g})$ -module in category \mathcal{O} , then the *twisted transfer matrix* is:

$$(9.35) \quad t_W(z; \varphi) := \sum_{\underline{i}, \underline{j} \in I^\ell} \left(\sum_{p \in \text{wt}(W)} \varphi(p) \times \text{Tr}_{W_p}(S_{\underline{i}\underline{j}}^W(z)) \right) E_{\underline{i}\underline{j}} \in \text{End}(\mathbf{V}^{\otimes \ell})[[z]].$$

If W is infinite-dimensional and the second summation above converges (for a generic choice of φ), then $t_W(z; \varphi)$ is still well-defined.

Lemma 9.4. *Let $i \in I_0$, $a, c \in \mathbb{C}^\times$ and $W := \mathcal{W}_{c,a}^{(i)}$. For $j, k \in I$, $f_{c,a}^{(i)}(z)s_{jk}^W(z)$ is a W -valued polynomial in z of degree ≤ 1 , where $f_{c,a}^{(i)}(z) = (1 - zaq^{M-N-i-1})$ if $i \leq M$, and $f_{c,a}^{(i)}(z) = \frac{(1 - zac^{-2}q^{M+N-i-1})(1 - zaq^{i-M-N-1})}{1 - zaq^{M+N-i-1}}$ if $i > M$.*

Proof. Set $V_m := W_{m, aq_i^{-1}}^{(i)} \otimes \mathbb{C}_{|m\varpi_i|}$ where \mathbb{C}_s for $s \in \mathbb{Z}_2$ denotes the one-dimensional parity module over $U_q(\widehat{\mathfrak{g}})$ of parity s . By [51, §4], $\mathcal{W}_{c,a}^{(i)}$ is an asymptotic limit ($q^m = c$ for $i \leq M$ and $q^m = c^{-1}$ otherwise) of the V_m for $m \rightarrow \infty$.

If $i \leq M$, then by Equation (2.23), $W_{m, aq_i^{-1}}^{(i)} \cong V_q^+(m\varpi_i; aq^{M-N-i-1})$. It follows from Equation (1.4) that $(1 - zaq^{M-N-i-1})s_{jk}^{V_m}(z)$ is a polynomial in z of degree ≤ 1 . Letting $m \rightarrow \infty$, we obtain similar statement for $\mathcal{W}_{c,a}^{(i)}$.

Suppose $i > M$. By comparing the highest ℓ -weights of the modules in Equation (2.24) based on (2.18), (2.20) and Lemma 2.6, we have:

$$\begin{aligned} W_{m, aq}^{(i)} &\cong V_q^{-*}(\lambda_m^{(i)}; aq^{M+N-1-i}) \cong \phi_{h_m}^*(z) \left(V_q^+(\lambda_m^{(i)}; aq^{i-M-N+2m-1}) \right), \\ h_m(z) &= \prod_{l=1}^m \prod_{j=1}^{M+N-i} \frac{(1 - zaq^{2l-2j+M+N-i-1})^2}{(1 - zaq^{2l-2j+M+N-i-3})(1 - zaq^{2l-2j+M+N-i+1})} \\ &= \frac{(1 - zaq^{2m+i-M-N-1})(1 - zaq^{-i+M+N-1})}{(1 - zaq^{2m-i+M+N-1})(1 - zaq^{i-M-N-1})}. \end{aligned}$$

It follows that $h_m(z)^{-1}(1 - zaq^{2m+i-M-N-1})s_{jk}^{V_m}(z)$ is a polynomial in z of degree ≤ 1 . By replacing q^m with c^{-1} in $h_m(z)^{-1}(1 - zaq^{2m+i-M-N-1})$, which results in $f_{c,a}^{(i)}(z)$, we obtain that $f_{c,a}^{(i)}(z)s_{jk}^{\mathcal{W}_{c,a}^{(i)}}(z)$ is also a polynomial in z of degree ≤ 1 . \square

Based on the lemma, let us define the $Y_q(\widehat{\mathfrak{g}})$ -module $\mathbb{W}_{c,a}^{(i)} := \phi_{f_{c,a}^{(i)}(z)}^*(\mathcal{W}_{c,a}^{(i)})$. (Indeed it can be equipped with a $U_q(\widehat{\mathfrak{g}})$ -module structure.)

Lemma 9.5. *For $i \in I_0$ and $a, c \in \mathbb{C}^\times$ we have:*

$$(9.36) \quad [\mathbb{W}_{c,1}^{(i)} \otimes \mathbb{W}_{1,a^2}^{(i)}] = [\mathbb{W}_{ca,a^2}^{(i)} \otimes \mathbb{W}_{a^{-1},1}^{(i)}] \in K_0(\mathcal{O}).$$

Let $X \in \mathcal{O}$ be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module. In the fractional ring of $K_0(\mathcal{O})$, $[X]$ is a sum of monomials in the $\frac{[\mathbb{W}_{b,a}^{(i)}]}{[\mathbb{W}_{c,a}^{(i)}]}[D]$ with $i \in I_0$, $a, b, c \in \mathbb{C}^\times$ and with $D \in \mathcal{O}$ one-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules, the number of monomials being $\dim X$.

Proof. For the first statement, by Definition 5.2 and Lemma 9.4 we have:

$$\omega_{c,1}^{(i)} \omega_{1,a^2}^{(i)} = \omega_{ca,a^2}^{(i)} \omega_{a^{-1},1}^{(i)}, \quad f_{c,1}^{(i)}(z) f_{1,a^2}^{(i)}(z) = f_{ca,a^2}^{(i)}(z) f_{a^{-1},1}^{(i)}(z).$$

Together with Proposition 5.3, this implies that the q -characters of the two tensor products in Equation (9.36) coincide. For the second statement, we argue as [23, Theorem 4.8] based on $\frac{\omega_{b,a}^{(i)}}{\omega_{c,a}^{(i)}} = \frac{\chi_q(\mathcal{W}_{b,a}^{(i)})}{\chi_q(\mathcal{W}_{c,a}^{(i)})} \equiv \frac{\chi_q(\mathbb{W}_{b,a}^{(i)})}{\chi_q(\mathbb{W}_{c,a}^{(i)})}$; see also [51, Corollary 8.5]. \square

Equation (9.36) is a *separation of variables* identity; see also [21, Theorem 3.11]. The same identity holds when replacing \mathbb{W} by \mathcal{W} . Since $t_{\mathbb{W}_{c,a}^{(i)}}(z)$ is a polynomial in z of degree $\leq \ell$, the following definition makes sense.

Definition 9.6. For $i \in I_0$ the *Baxter operator* is $Q_i(z) := t_{\mathbb{W}_{z,1}^{(i)}}(1)$.

Let $p_c^{(i)} = \varpi(\omega_{c,a}^{(i)})$. Then $\text{wt}(\mathbb{W}_{c,a}^{(i)}) \subset p_c^{(i)} q^{\mathbf{Q}^-}$ and $\overline{Q}_i(z) := (p_c^{(i)})^{-1} Q_i(z)$ is a power series in the $q^{-\alpha_j}$ with $j \in I_0$ whose coefficients are in $\text{End}(\mathbf{V}^{\otimes \ell})[z, z^{-1}]$. Let $\overline{Q}_i^0(z)$ be its leading term. Since $(\mathbb{W}_{1,1}^{(i)})_{p_1^{(i)}}$ is the one-dimensional simple socle

of $\mathbb{W}_{1,1}^{(i)}$, by Definition 9.1, \underline{i} is an eigenvector of $\overline{Q}_i^0(1)$ with non-zero eigenvalue. (Here we used the overall assumption $b_l \notin q^{\mathbb{Z}}$.) The formal power series $\overline{Q}_i^0(z)$ and $Q_i(z)$ in the $q^{-\alpha_j}$ can therefore be inverted for $z \in \mathbb{C}$ generic.

Corollary 9.7 (generalized Baxter TQ relations). *For $b, c \in \mathbb{C}^\times$, we have:*

$$(9.37) \quad \frac{t_{\mathbb{W}_{b,1}^{(i)}}(z^{-2})}{t_{\mathbb{W}_{c,1}^{(i)}}(z^{-2})} = \frac{Q_i(zb)}{Q_i(zc)}, \quad \frac{t_{\mathbb{W}_{b,1}^{(i)}}(z^{-2})}{t_{\mathbb{W}_{c,1}^{(i)}}(z^{-2})} = \prod_{l=1}^{\ell} \frac{f_{c,1}^{(i)}(z^{-2}b_l^{-2})}{f_{b,1}^{(i)}(z^{-2}b_l^{-2})} \times \frac{Q_i(zb)}{Q_i(zc)}.$$

If $X \in \mathcal{O}$ is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module, then $t_X(z^{-2})$ is a sum of monomials in the $\frac{Q_i(zb)}{Q_i(zc)} t_D(z^{-2})$ with $i \in I_0$, $b, c \in \mathbb{C}^\times$ and with $D \in \mathcal{O}$ one-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules, the number of terms being $\dim X$.

Proof. In Equation (9.36) let us set $(a, c) = (z^{-1}, bz)$:

$$[\mathbb{W}_{b,z^{-2}}^{(i)}][\mathbb{W}_{z,1}^{(i)}] = [\mathbb{W}_{zb,1}^{(i)}][\mathbb{W}_{1,z^{-2}}^{(i)}].$$

Taking transfer matrices and evaluating them at 1 gives the special case $c = 1$ of Equation (9.37), which in turn implies the general case $c \in \mathbb{C}^\times$. The second statement is a translation of that of Lemma 9.5. \square

Example 9.8. Let $\mathfrak{g} = \mathfrak{gl}(2|2)$ and $X = W_{1,1}^{(1)} = V_q^+(\epsilon_1; q^{-1})$. By Equation (2.17):

$$\chi_q(X) = \boxed{1}_1 + \boxed{2}_1 + \boxed{3}_1 + \boxed{4}_1.$$

If $s \in \mathbb{Z}_2$, $g(z) \in \mathbb{C}[[z]]^\times$ and $c \in \mathbb{C}^\times$, for simplicity let $sg(z) := (g(z)^4; s) \in \widehat{\mathfrak{P}}$, $[s, g(z)] := [L(g(z)^4; s)] \in K_0(\mathcal{O})$ and $\langle s, c \rangle := (c^4; s) \in \mathfrak{P}$. Set $w_{c,a}^{(i)} := f_{c,a}^{(i)}(z) \omega_{c,a}^{(i)}$. By Definitions 2.2, 5.2 and Lemma 9.4:

$$\begin{aligned} \boxed{1}_1 &= \left(\frac{q-z}{1-zq}, 1, 1, 1; \overline{0} \right), \quad \boxed{2}_1 = \left(1, \frac{q-zq^2}{1-zq^3}, 1, 1; \overline{0} \right), \\ \boxed{3}_1 &= \left(1, 1, \frac{1-zq^3}{q-zq^2}, 1; \overline{1} \right), \quad \boxed{4}_1 = \left(1, 1, 1, \frac{1-zq}{q-z}; \overline{1} \right), \\ \frac{w_{c,a}^{(1)}}{w_{1,a}^{(1)}} &= \left(\frac{c-zac^{-1}}{1-za}, 1, 1, 1; \overline{0} \right), \quad \frac{w_{c,a}^{(2)}}{w_{1,a}^{(2)}} = \left(\frac{c-zaqc^{-1}}{1-zaq}, \frac{c-zaqc^{-1}}{1-zaq}, 1, 1; \overline{0} \right), \\ \frac{w_{c,a}^{(3)}}{w_{1,a}^{(3)}} &= \left(\frac{1-zac^{-2}}{1-za}, \frac{1-zac^{-2}}{1-za}, \frac{1-zac^{-2}}{1-za}, c^{-1}; \overline{0} \right), \quad \boxed{1}_1 = \frac{w_{q,q}^{(1)}}{w_{1,q}^{(1)}}, \\ \boxed{2}_1 &= \frac{w_{q^{-1},q}^{(1)}}{w_{1,q}^{(1)}} \frac{w_{q,q^2}^{(2)}}{w_{1,q^2}^{(2)}}, \quad \boxed{3}_1 = \overline{1} q^{-1} \frac{w_{q,q^2}^{(2)}}{w_{1,q^2}^{(2)}} \frac{w_{q^{-1},q}^{(3)}}{w_{1,q}^{(3)}}, \quad \boxed{4}_1 = \overline{1} \frac{1-zq}{1-zq^{-1}} \frac{w_{q,q}^{(3)}}{w_{1,q}^{(3)}}. \end{aligned}$$

It follows that in the fractional ring of $K_0(\mathcal{O})$:

$$[X] = \frac{[\mathbb{W}_{q,q}^{(1)}]}{[\mathbb{W}_{1,q}^{(1)}]} + \frac{[\mathbb{W}_{q^{-1},q}^{(1)}]}{[\mathbb{W}_{1,q}^{(1)}]} \frac{[\mathbb{W}_{q,q^2}^{(2)}]}{[\mathbb{W}_{1,q^2}^{(2)}]} + [\overline{1}, q^{-1}] \frac{[\mathbb{W}_{q,q^2}^{(2)}]}{[\mathbb{W}_{1,q^2}^{(2)}]} \frac{[\mathbb{W}_{q^{-1},q}^{(3)}]}{[\mathbb{W}_{1,q}^{(3)}]} + [\overline{1}, \frac{1-zq}{1-zq^{-1}}] \frac{[\mathbb{W}_{q,q}^{(3)}]}{[\mathbb{W}_{1,q}^{(3)}]}.$$

Let $q^{\frac{1}{2}}$ be a square root of q . By Example 9.2 and Equation (9.37):

$$\begin{aligned} t_X(z^{-2}) &= \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \frac{Q_1(zq^{-\frac{3}{2}})}{Q_1(zq^{-\frac{1}{2}})} \frac{Q_2(z)}{Q_2(zq^{-1})} + \langle \overline{1}, q^{-1} \rangle \times \frac{Q_2(z)}{Q_2(zq^{-1})} \frac{Q_3(zq^{-\frac{3}{2}})}{Q_3(zq^{-\frac{1}{2}})} q^{-\ell} \\ &\quad + \langle \overline{1}, 1 \rangle \times \frac{Q_3(zq^{\frac{1}{2}})}{Q_3(zq^{-\frac{1}{2}})} \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 - b_l q^{-1}}. \end{aligned}$$

Example 9.9. Let $\mathfrak{g} = \mathfrak{gl}(2|0)$ and $X = W_{1,1}^{(1)} = V_q^+(\epsilon_1; q)$. Then

$$\begin{aligned} \boxed{1}_{q^2} + \boxed{2}_{q^2} &= \left(\frac{q - zq^{-2}}{1 - zq^{-1}}, 1; \bar{0} \right) + \left(1, \frac{q - z}{1 - zq}; \bar{0} \right) = \frac{w_{q,q}^{(1)}}{w_{1,q}^{(1)}} + \frac{q - z}{1 - zq} \frac{w_{q^{-1},q}^{(1)}}{w_{1,q}^{(1)}}, \\ t_X(z^{-2}) &= \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \langle \bar{0}, q \rangle \times \frac{Q_1(zq^{-\frac{3}{2}})}{Q_1(zq^{-\frac{1}{2}})} \prod_{l=1}^{\ell} \frac{qz^2 - b_l}{z^2 - b_l q}. \end{aligned}$$

Example 9.10. Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $X = W_{1,1}^{(1)} = V_q^+(\epsilon_1; q^{-1})$. We have

$$\begin{aligned} \chi_q(X) = \boxed{1}_1 + \boxed{2}_1 &= \left(\frac{q - z}{1 - zq}, 1; \bar{0} \right) + \left(1, \frac{1 - zq}{q - z}; \bar{1} \right) = \frac{w_{q,q}^{(1)}}{w_{1,q}^{(1)}} \left(1 + \bar{1} \frac{1 - zq}{q - z} \right), \\ t_X(z^{-2}) &= \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \langle \bar{1}, q^{-1} \rangle \times \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 q - b_l}. \end{aligned}$$

One can view Examples 9.9 and 9.10 as degenerate cases of Example 9.8.

We are ready to deduce three-term functional relations of the Baxter operators $Q_i(z)$. Fix $a = 1$ and $c, d \in \mathbb{C}^\times \setminus q^{\mathbb{Z}}$. In Equation (5.30) let us evaluate transfer matrices at z^{-2} making use of Proposition 9.3:

$$\begin{aligned} t_{M_{c,1}}^{(i)}(z^{-2}) t_{\mathcal{W}_{d,d^2}}^{(i)}(z^{-2}) &= t_{\mathcal{W}_{d,q_i,d^2}}^{(i)}(z^{-2}) \prod_{j \in I_0: j \sim i} t_{\mathcal{W}_{c_{ij}^{-1}, q_{ij}^{-1}, c_{ij}^{-2}}}^{(j)}(z^{-2}) \\ &\quad + t_{D_i}(z^{-2}) t_{\mathcal{W}_{d\hat{q}_i^{-1}, d^2}}^{(i)}(z^{-2}) \prod_{j \in I_0: j \sim i} t_{\mathcal{W}_{c_{ij}^{-1}, q_{ij}^{-1}, c_{ij}^{-2}}}^{(j)}(z^{-2}). \end{aligned}$$

Dividing both sides by the term at the second row without $t_{D_i}(z^{-2})$ and making use of Equation (9.37), we obtain the *Baxter TQ relation*:

$$(9.38) \quad X_c^{(i)}(z) \frac{Q_i(z)}{Q_i(z\hat{q}_i^{-1})} = y_i(z) \frac{Q_i(zq_i)}{Q_i(z\hat{q}_i^{-1})} \prod_{j \in I_0: j \sim i} \frac{Q_j(zq_{ij}^{\frac{1}{2}})}{Q_j(zq_{ij}^{-\frac{1}{2}})} + t_{D_i}(z^{-2}),$$

where $X_c^{(i)}(z)$ (depending on $c \in \mathbb{C}^\times \setminus q^{\mathbb{Z}}$) and $y_i(z)$ are given by

$$\begin{aligned} X_c^{(i)}(z) &= \frac{t_{M_{c,1}}^{(i)}(z^{-2})}{\prod_{j \in I_0: j \sim i} t_{\mathcal{W}_{c_{ij}^{-1}, q_{ij}^{-1}, c_{ij}^{-2}}}^{(j)}(z^{-2})} \times \prod_{l=1}^{\ell} \frac{f_{d\hat{q}_i^{-1}, d^2}^{(i)}(z^{-2}b_l)}{f_{d, d^2}^{(i)}(z^{-2}b_l)}, \\ y_i(z) &= \prod_{l=1}^{\ell} \left(\frac{f_{d\hat{q}_i^{-1}, d^2}^{(i)}(z^{-2}b_l)}{f_{d, d^2}^{(i)}(z^{-2}b_l)} \times \prod_{j \in I_0: j \sim i} \frac{f_{c_{ij}^{-1}, q_{ij}^{-1}, c_{ij}^{-2}}^{(j)}(z^{-2}b_l)}{f_{c_{ij}^{-1}, q_{ij}^{-1}, c_{ij}^{-2}}^{(j)}(z^{-2}b_l)} \right). \end{aligned}$$

Note that $y_i(z), D_i$ are independent of c, d by Lemma 9.4 and Theorem 5.5.

Let us assume that the twisted transfer matrices in Equation (9.35) are well-defined for all the $M_{c,1}^{(i)}$ and $\mathcal{W}_{c,a}^{(i)}$, upon a generic choice of $\varphi : \mathfrak{P} \rightarrow \mathbb{C}^\times$; this corresponds to the convergence assumption in [23, Remark 5.12 (ii)]. Then Equation (9.38) is an operator equation in $\text{End}(\mathbf{V}^{\otimes \ell})[[z^{-2}]]$.

Based on the asymptotic construction of $\mathcal{W}_{c,a}^{(i)}$, one can show that there exists $n \in \mathbb{Z}$ such that $z^n Q_i(z)$ is a polynomial in z with values in $\text{End}(\mathbf{V}^{\otimes \ell})$.

As in [24, §5], we expect that the $t_{M_{c,1}}^{(i)}(z^{-2})$ are polynomials in z^{-2} (up to multiplication by an integer power of z). Suppose that w is a zero of $Q_i(z)$ that is neither a zero of $Q_i(z\hat{q}_i^{-1}), Q_j(zq_{ij}^{-\frac{1}{2}})$ nor a pole of $X_c^{(i)}(z)$. Then we have the

Bethe Ansatz Equation: (see [42, (2.6a)] and [5, 36])

$$(9.39) \quad y_i(w) \frac{Q_i(wq_i)}{Q_i(w\hat{q}_i^{-1})} \prod_{j \in I_0: j \sim i} \frac{Q_j(wq_{ij}^{\frac{1}{2}})}{Q_j(wq_{ij}^{-\frac{1}{2}})} = -t_{D_i}(w^{-2}).$$

Example 9.11. Following Example 9.8, let us describe the highest ℓ -weight (still denoted by D_i) of the one-dimensional $U_q(\widehat{\mathfrak{g}})$ -module D_i and $y_i(z)$ in Equation (9.39) for $\mathfrak{g} = \mathfrak{gl}(2|2)$. First compute the $\omega_{c,a}^{(i)}, A_{i,a}$ by Definitions 5.2 and 2.2:

$$\begin{aligned} \omega_{c,a}^{(1)} &= \left(\frac{c - zac^{-1}}{1 - za}, 1, 1, 1; \bar{0} \right), \quad \omega_{c,a}^{(2)} = \left(\frac{c - zaqc^{-1}}{1 - zaq}, \frac{c - zaqc^{-1}}{1 - zaq}, 1, 1; \bar{0} \right), \\ \omega_{c,a}^{(3)} &= \left(1, 1, 1, \frac{1 - za}{c - zac^{-1}}; \bar{0} \right), \quad A_{1,a} = \left(\frac{q - zaq^{-1}}{1 - za}, \frac{1 - zaq^2}{q - zaq}, 1, 1; \bar{0} \right), \\ A_{2,a} &= \left(1, \frac{q - za}{1 - zaq}, \frac{q - za}{1 - zaq}, 1; \bar{1} \right), \quad A_{3,a} = \left(1, 1, \frac{1 - zaq^2}{q - zaq}, \frac{q - zaq^{-1}}{1 - za}; \bar{0} \right). \end{aligned}$$

The relations between A and ω are as follows: $A_{1,a} = \omega_{q^2, aq^2}^{(1)} \omega_{q^{-1}, aq^{-1}}^{(2)}$ and

$$A_{2,a} = \bar{1} \frac{q - za}{1 - zaq} \omega_{q^{-1}, aq^{-1}}^{(1)} \omega_{q, aq}^{(3)}, \quad A_{3,a} = \frac{1 - zaq^2}{q - zaq} \omega_{q, aq}^{(2)} \omega_{q^{-2}, aq^{-2}}^{(3)}.$$

It follows that $D_1 = 1$, $D_2 = \bar{1} \frac{1 - zq}{q - z}$, $D_3 = \frac{q - zq}{1 - zq^2}$ and so $(D_i(z) := t_{D_i}(z^{-2}))$

$$\begin{aligned} D_1(z) &= 1, \quad D_2(z) = \langle \bar{1}, q^{-1} \rangle \times \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 q - b_l}, \quad D_3(z) = \langle \bar{0}, q \rangle \times \prod_{l=1}^{\ell} \frac{z^2 q - b_l q}{z^2 - b_l q^2}, \\ y_1(z) &= 1, \quad y_2(z) = \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 - b_l q^{-1}}, \quad y_3(z) = \prod_{l=1}^{\ell} \frac{z^2 - b_l q^{-2}}{z^2 - b_l q^2}. \end{aligned}$$

The Bethe Ansatz Equations become in this case:

$$\begin{aligned} \frac{Q_1(w_1 q)}{Q_1(w_1 q^{-1})} \frac{Q_2(w_1 q^{-\frac{1}{2}})}{Q_2(w_1 q^{\frac{1}{2}})} &= -1, \quad \frac{Q_1(w_2 q^{-\frac{1}{2}})}{Q_1(w_2 q^{\frac{1}{2}})} \frac{Q_3(w_2 q^{\frac{1}{2}})}{Q_3(w_2 q^{-\frac{1}{2}})} = -\langle \bar{1}, q^{-1} \rangle \times q^{-\ell}, \\ \frac{Q_3(w_3 q^{-1})}{Q_3(w_3 q)} \frac{Q_2(w_3 q^{\frac{1}{2}})}{Q_2(w_3 q^{-\frac{1}{2}})} &= -\langle \bar{0}, q \rangle \times \prod_{l=1}^{\ell} \frac{w_3^2 q - b_l q}{w_3^2 - b_l q^{-2}}, \end{aligned}$$

where w_i is a zero of $Q_i(z)$ for $1 \leq i \leq 3$.

The generalized Baxter relations in Lemma 9.5 and Bethe Ansatz Equations (9.39) for the Baxter operators $Q_i(z)$ are based on asymptotic $U_q(\widehat{\mathfrak{g}})$ -modules: $\mathcal{W}_{c,a}^{(i)}, \mathcal{N}_{c,a}^{(i)}, M_{c,a}^{(i)}$, whereas in recent parallel works [17, 33, 24, 18] representations of Borel subalgebras $(Y_q(\mathfrak{g}))$ in our situation) play a key rôle.

In [5, 36], for the Yangian of $\mathfrak{gl}(M|N)$ the Baxter operators $\mathbf{Q}_J(z)$ are labeled by the subsets J of I . In addition to TQ relations, there are algebraic relations among the $\mathbf{Q}_J(z)$ called QQ relations. Our $Q_i(z)$ with $i \in I_0$ seem to be algebraically independent by Proposition 7.7; see also [23, Theorem 4.11].

Remark 9.12. Following [6, 23] define $\mathbf{Q}_i(z) := t_{L_{i,1}^+}(z)$ for $i \in I_0$. We have

$$(9.40) \quad t_{L([c]_i)}(z^{-2}) \frac{\mathbf{Q}_i(z^{-2} c^{-2})}{\mathbf{Q}_i(z^{-2})} = \prod_{l=1}^{\ell} \frac{f_{1,1}^{(i)}(z^{-2} b_l^{-2})}{f_{c,1}^{(i)}(z^{-2} b_l^{-2})} \times \frac{Q_i(zc)}{Q_i(z)}$$

based on the q -character formula $\frac{\chi_q(\mathcal{W}_{c,1}^{(i)})}{\chi_q(\mathcal{W}_{1,1}^{(i)})} = [c]_i \frac{\chi_q(L_{i,c-2}^+)}{\chi_q(L_{i,1}^+)}$ and Equation (9.37). See [21, Remark A.7] for a similar comparison in the Yangian case.

Acknowledgments. The author thanks Vyjayanthi Chari, Giovanni Felder, David Hernandez and Marc Rosso for enlightening discussions. He is supported by the National Center of Competence in Research SwissMAP—The Mathematics of Physics of the Swiss National Science Foundation.

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